Signals and Systems

A Primer with MATLAB®

SECOND EDITION

Matthew N. O. Sadiku
Warsame H. Ali
Sarhan M. Musa

CRC Press
Taylor & Francis Group

Signals and Systems

The idea of signals and systems arises in different disciplines such as science, engineering, economics, politics, and medicine. Typical examples of systems include radio and television, telephone networks, radar systems, computer networks, wireless communication, military surveillance systems, and satellite communication systems. Knowledge of a broad range of signals and systems is of practical value to engineering students because engineers must be familiar with signal and system concepts to analyze some specific signals and systems they will deal with in their professional lives. This book is intended to serve as a textbook for junior-level students in electrical and computer engineering. The main aim of this second edition is to improve the clarity of the first edition and fulfill the same objective as the first, which is to present continuous-time and discrete-time signals and systems to electrical and computer engineering students in a manner that is clearer, more interesting, and easier to understand than other texts.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Signals and Systems
A Primer with MATLAB®

Second Edition

Matthew N. O. Sadiku, Warsame H. Ali,
and Sarhan M. Musa

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

MATLAB® and Simulink® are trademarks of The MathWorks, Inc. and are used with permission. The MathWorks does not warrant the accuracy of the text or exercises in this book. This book's use or discussion of MATLAB® or Simulink® software or related products does not constitute endorsement or sponsorship by The MathWorks of a particular pedagogical approach or particular use of the MATLAB® and Simulink® software.

Cover: Web Large Image (Public)

Second edition published 2024
by CRC Press
2385 NW Executive Center Drive, Suite 320, Boca Raton FL 33431

and by CRC Press
4 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

CRC Press is an imprint of Taylor & Francis Group, LLC

© 2024 [Matthew N. O. Sadiku, Warsame H. Ali, and Sarhan M. Musa]

First edition published by CRC Press 2016

Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, access www.copyright.com or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. For works that are not available on CCC please contact mpkbookspermissions@tandf.co.uk

Trademark Notice: Product or corporate names may be trademarks or registered trademarks and are used only for identification and explanation without intent to infringe.

ISBN: 978-1-032-46867-9 (hbk)
ISBN: 978-1-032-79176-0 (pbk)
ISBN: 978-1-003-49086-9 (ebk)

DOI: 10.1201/9781003490869

Typeset in Times LT Std
by KnowledgeWorks Global Ltd.

Access the Instructor and Student Resources: www.routledge.com/9781032468679

My parents, Ayisat and Solomon Sadiku, and my wife, Janet

Matthew N. O. Sadiku

My parents, Faduma Ibrahim Omer and Hassan Ali Hussein (Shigshigow), for their unconditional love and support; to my beloved wife, engineer Shukri Mahdi Ali; and my children, Mohamed, Faduma, Dahaba, Hassan, and Khadid, for their patience, and to all my siblings

Warsame H. Ali

My parents, Mahmoud Musa and Fatmeh Hussein, and my wife, Lama

Sarhan M. Musa

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Contents

Preface xiii
Author Biographies xv
Notes to Students xvii

Chapter 1 Basic Concepts 1

Global Positioning System 1
1.1 Introduction 2
1.2 Basic Definitions 2
1.3 Classification of Signals 3
1.3.1 Continuous-Time and Discrete-Time Signals 3
1.3.2 Periodic and Nonperiodic Signals 4
1.3.3 Analog and Digital Signals 6
1.3.4 Energy and Power Signals 7
1.3.5 Even and Odd Symmetry 8
1.4 Basic Continous-Time Signals 13
1.4.1 Unit Step Function 13
1.4.2 Unit Impulse Function 15
1.4.3 Unit Ramp Function 17
1.4.4 Rectangular Pulse Function 18
1.4.5 Triangular Pulse Function 19
1.4.6 Sinusoidal Signal 20
1.4.7 Exponential Signal 20
1.5 Basic Discrete-Time Signals 27
1.5.1 The Unit Step Sequence 27
1.5.2 The Unit Impulse Sequence 28
1.5.3 Unit Ramp Sequence 28
1.5.4 Sinusoidal Sequence 29
1.5.5 Exponential Sequence 30
1.6 Basic Operations on Continous Signals 32
1.6.1 Time Reversal 32
1.6.2 Time Scaling 33
1.6.3 Time Shifting 34
1.6.4 Amplitude Transformations 34
1.7 Basic Operations on Discrete Signals 36
1.7.1 Time Shifting 38
1.7.2 Time Reversal 38
1.7.3 Expansion 38
1.8 Classification of Systems 43
1.8.1 Continuous-Time and Discrete-Time Systems 46
1.8.2 Causal and Noncausal Systems 46
1.8.3 Linear and Nonlinear Systems 47

1.8.4 Time-Varying and Time-Invariant Systems 48
1.8.5 Systems With and Without Memory 50
1.9 Applications 52
1.9.1 Electric Circuits 52
1.9.2 Square-Law Devices 52
1.9.3 DSP Systems 53
1.10 Computing With MATLAB® 54
Summary 58
Review Questions 59
Problems 60

Chapter 2 Convolution 74

Enhancing Your Communication Skills 74
2.1 Introduction 75
2.2 Impulse Response 75
2.3 The Convolution Integral 76
2.4 Graphical Convolution 81
2.5 Block Diagram Representation 88
2.6 Discrete-Time Convolution 90
2.7 Block Diagram Realization 98
2.8 Deconvolution 99
2.9 Computing With MATLAB® 101
2.10 Applications 104
2.10.1 BIBO Stability of Continuous-Time Systems 104
2.10.2 BIBO Stability of Discrete-Time Systems 105
2.10.3 Circuit Analysis 106
Summary 108
Review Questions 109
Problems 110

Chapter 3 The Laplace Transform 123

Historical Profile 123
3.1 Introduction 124
3.2 Definition of the Laplace Transform 124
3.3 Properties of the Laplace Transform 128
3.3.1 Linearity 128
3.3.2 Scaling 129
3.3.3 Time Shifting 130
3.3.4 Frequency Shifting 131
3.3.5 Time Differentiation 131
3.3.6 Time Convolution 132
3.3.7 Time Integration 133

3.3.8 Frequency Differentiation 134
3.3.9 Time Periodicity 135
3.3.10 Modulation 136
3.3.11 Initial and Final Values 137
3.4 The Inverse Laplace Transform 144
3.4.1 Simple Poles 145
3.4.2 Repeated Poles 146
3.4.3 Complex Poles 147
3.5 Transfer Function 155
3.6 Applications 160
3.6.1 Integrodifferential Equations 160
3.6.2 Circuit Analysis 163
3.6.3 Control Systems 167
3.7 Computing With MATLAB® 169
Summary 176
Review Questions 176
Problems 177

Chapter 4 The Fourier Series 190

Historical Profile 190
4.1 Introduction 191
4.2 Trigonometric Fourier Series 192
4.3 Exponential Fourier Series 199
4.4 Properties of Fourier Series 205
4.4.1 Linearity 205
4.4.2 Time Shifting 206
4.4.3 Time Reversal 207
4.4.4 Time Scaling 207
4.4.5 Even and Odd Symmetries 208
4.4.6 Parseval's Theorem 209
4.5 Truncated Complex Fourier Series 214
4.6 Applications 216
4.6.1 Circuit Analysis 216
4.6.2 Spectrum Analyzers 217
4.6.3 Filters 218
4.7 Computing With MATLAB® 221
Summary 227
Review Questions 228
Problems 229

Chapter 5 The Fourier Transform 243

5.1 Introduction 244
5.2 Definition of the Fourier Transform 244

5.3 Properties of Fourier Transform 252
5.3.1 Linearity 252
5.3.2 Time Scaling 252
5.3.3 Time Shifting 253
5.3.4 Frequency Shifting 254
5.3.5 Time Differentiation 254
5.3.6 Frequency Differentiation 255
5.3.7 Time Integration 255
5.3.8 Duality 256
5.3.9 Convolution 257
5.4 Inverse Fourier Transform 262
5.5 Applications 263
5.5.1 Circuit Analysis 264
5.5.2 Amplitude Modulation 267
5.5.3 Sampling 270
5.6 Parseval's Theorem 273
5.7 Comparing the Fourier and Laplace Transforms 276
5.8 Computing With MATLAB® 277
Summary 280
Review Questions 281
Problems 282

Chapter 6 Discrete Fourier Transform 295

Campus Communication Network 295
6.1 Introduction 296
6.2 Discrete-Time Fourier Transform 296
6.3 Properties of DTFT 301
6.3.1 Linearity 301
6.3.2 Time Shifting and Frequency Shifting 302
6.3.3 Time Reversal and Conjugation 303
6.3.4 Time Scaling 304
6.3.5 Frequency Differentiation 305
6.3.6 Time and Frequency Convolution 306
6.3.7 Accumulation 307
6.3.8 Parseval's Relation 308
6.4 Discrete Fourier Transform 313
6.5 Fast Fourier Transform 319
6.6 Computing With MATLAB® 320
6.7 Applications 323
6.7.1 Touch-Tone Telephone 323
6.7.2 Windowing 324
Summary 327
Review Questions 327
Problems 328

Chapter 7 z-Transform 336

Codes of Ethics 336
7.1 Introduction 337
7.2 Definition of the z-Transform 337
7.3 Region of Convergence 341
7.4 Properties of the z-Transform 343
7.4.1 Linearity 344
7.4.2 Time Shifting 344
7.4.3 Frequency Scaling 345
7.4.4 Time Reversal 346
7.4.5 Modulation 346
7.4.6 Accumulation 347
7.4.7 Convolution 348
7.4.8 Initial and Final Values 348
7.5 Inverse z-Transform 355
7.5.1 Long Division Expansion 355
7.5.2 Partial Fraction Expansion 357
7.6 Applications 360
7.6.1 Linear Difference Equation 361
7.6.2 Transfer Function 363
7.7 Computing With MATLAB® 367
Summary 371
Review Questions 372
Problems 373

Appendix A: Mathematical Formulas 383

Appendix B: Complex Numbers 395

Appendix C: MATLAB® 403

Appendix D: Answers to Odd-Numbered Problems 417

Index 449

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Preface

The main objective of this second edition is the same as the first edition: to present continuous-time and discrete-time signals and systems to electrical and computer engineering students in a manner that is clearer, more interesting, and easier to understand than other texts. This objective is achieved in the following ways:

- We have included several features to help students feel at home with the subject. Each chapter opens with a historical profile or career talk. This is followed by an introduction that links the chapter with the previous chapters and states the chapter objectives. The chapter ends with a summary of key points and formulas.
- All principles are presented in a lucid, logical, step-by-step manner. As much as possible, we avoid wordiness and giving too much detail that could hide concepts and impede overall understanding of the material.
- Important formulas are boxed as a means of helping students sort out what is essential from what is not. Also, to ensure that students clearly get the gist of the matter, key terms are defined and highlighted.
- Thoroughly worked examples are liberally given at the end of every section. The examples are regarded as a part of the text and are clearly explained without asking the reader to fill in the missing steps. Thoroughly worked-out examples give students a good understanding of the solution and the confidence to solve problems themselves. Some of the problems are solved in two or three ways to facilitate an understanding and comparison of different approaches.
- To give students practice opportunity, each illustrative example is immediately followed by a practice problem with the answer. The students can follow the example step by step to solve the practice problem without flipping pages or looking at the end of the book for answers. The practice problem is also intended to test if students understand the preceding example. It will reinforce their grasp of the material before they move on to the next section.
- Toward the end of each chapter, we discuss some application aspects of the concepts covered in the chapter. The material covered in the chapter is applied to at least one or two practical problems or devices. This helps the students see how the concepts are applied to real-life situations.
- Ten review questions in the form of multiple-choice objective items are provided at the end of each chapter with answers. The review questions are intended to cover the little "tricks" which the examples and end-of-chapter problems may not cover. They serve as a self-test device and help students determine how well they have mastered the chapter.

The idea of signals and systems arises in different disciplines such as science, engineering, economics, politics, and medicine. Most students in electrical and computer engineering will deal with signals and systems in their professional lives.

This is why a course on signals and systems is an important part of most engineering curricula.

Some books on signals and systems are designed for a two-semester course sequence. Unfortunately, electrical engineering has changed considerably, and its curriculum is so crowded that there is no room for a two-semester course on signals and systems. This book is designed for a three-hour semester course on signals and systems. It is intended as a textbook for junior-level students in electrical and computer engineering curricula. The prerequisites for a course based on this book are standard mathematics (including calculus and differential equations) and electric circuit analysis.

In recognition of the requirements set by the Accreditation Board for Engineering and Technology (ABET) on integrating computer tools, the use of MATLAB® is encouraged in a student-friendly manner. We have introduced MATLAB in Appendix B and applied it gradually throughout this book. MATLAB has become a standard software package in electrical engineering curricula. To avoid confusing learning MATLAB with the main thing (signals and systems), we have delayed MATLAB to the end of each chapter.

The main changes in the second edition are made to improve the clarity of the text. This is achieved by rewriting some sections, replacing some examples and corresponding practice problems with better ones, adding new problems, and deleting or replacing some problems.

A solutions manual is also available on the Routledge website (www.routledge.com/9781032468679).

We are grateful for the support of Dr. Annamalia Annamalai, the department head of the Department of Electrical and Computer Engineering, and Dr. Pamela Obiomon, the dean of the College of Engineering at Prairie View A&M University, Prairie View, Texas. Finally, we want to express our profound gratitude to Janet (Matthew's wife) for contributing in many ways.

M. N. O. Sadiku, W. Ali, S. M. Musa

Author Biographies

Matthew N. O. Sadiku received his BSc degree in 1978 from Ahmadu Bello University, Zaria, Nigeria, and his MSc and PhD degrees from Tennessee Technological University, Cookeville, Tennessee, in 1982 and 1984, respectively. From 1984 to 1988, he was an assistant professor at Florida Atlantic University, Boca Raton, Florida, where he did graduate work in computer science. From 1988 to 2000, he was at Temple University, Philadelphia, Pennsylvania, where he became a full professor. From 2000 to 2002, he was with Lucent/Avaya, Holmdel, New Jersey, as a system engineer and with Boeing Satellite Systems, Los Angeles, California, as a senior scientist. He is presently a professor emeritus of electrical and computer engineering at Prairie View A&M University, Prairie View, Texas.

He is the author of over 1,200 professional papers and over 130 books, including *Elements of Electromagnetics* (Oxford University Press, 7th ed., 2018), *Fundamentals of Electric Circuits* (McGraw-Hill, 7th ed., 2021, with C. Alexander), *Computational Electromagnetics with MATLAB®* (CRC Press, 4th ed., 2019), *Principles of Modern Communication Systems* (Cambridge University Press, 2017, with S.O. Agbo), and *Emerging Internet-Based Technologies* (CRC Press, 2019). In addition to the engineering books, he has written Christian books, including *Secrets of Successful Marriages*, *How to Discover God's Will for Your Life*, and commentaries on all the books of the New Testament Bible. Some of his books have been translated into French, Korean, Chinese (and Chinese Long Form in Taiwan), Italian, Portuguese, and Spanish.

He was the recipient of the 2000 McGraw-Hill/Jacob Millman Award for outstanding contributions in the field of electrical engineering. He was also the recipient of the Regents Professor Award for 2012–2013 by the Texas A&M University System. He is a registered professional engineer and a life fellow of the Institute of Electrical and Electronics Engineers (IEEE) "for contributions to computational electromagnetics and engineering education." He was the IEEE Region 2 Student Activities Committee Chairman. He was an associate editor for *IEEE Transactions on Education*. He is also a member of the Association for Computing Machinery (ACM) and the American Society of Engineering Education (ASEE). His current research interests are in the areas of computational electromagnetics, computer networks, and engineering education. His works can be found in his autobiography, *My Life and Work* (Trafford Publishing, 2017) or his website www.matthew-sadiku.com. He currently resides in West Palm Beach, Florida, with his wife Janet. He can be reached via email at sadiku@ieee.org

Warsame H. Ali received his BSc from King Saud University Electrical Engineering Department, Riyadh, Saudi Arabia, and his MS from Prairie View A&M University, Prairie View, Texas. He received his PhD in electrical engineering from the University of Houston, Houston, Texas. Dr. Ali was promoted to associate professor and tenured in 2010 and promoted to full professor in 2017. Dr. Ali joined NASA, Glenn Research Center, in the summer of 2005, and Texas Instruments (TI) in 2006 as a faculty fellow.

Dr. Ali has given several invited talks and is also the author of more than 100 research articles in major scientific journals and conferences. Dr. Ali has received several major National Science Foundation (NSF), Naval Sea Systems Command (NAVSEA), Air Force Research Laboratory (AFRL), and Department of Energy (DOE) awards. At present, he is teaching undergraduate and graduate courses in the Electrical and Computer Engineering Department at Prairie View A&M University. His main research interests are the application of digital PID controllers, digital methods to electrical measurements, and mixed-signal testing techniques, power systems, high voltage direct current (HVDC) power transmission, sustainable power and energy systems, power electronics and motor drives, electric and hybrid vehicles, and control systems.

Sarhan M. Musa, is a professor in the Electrical and Computer Engineering Department at Prairie View A&M University. He holds a PhD in electrical engineering from the City University of New York. He is the founder and director of Prairie View Networking Academy (PVNA), Texas. He is an LTD Sprint and Boeing Welliver Fellow. Professor Musa is internationally known through his research, scholarly work, and published books. He has had a number of invited talks at international conferences. He has received a number of prestigious national and university awards and research grants. He is a senior member of the IEEE and has also served as a member of the technical program committee and steering committee for a number of major journals and conferences. Professor Musa has written more than a dozen books on various areas in electrical and computer engineering. His current research interests cover many topics in artificial intelligence/machine learning, renewable energy, power system, control systems, and computational methods.

Notes to Students

Although electrical engineering is an exciting and challenging discipline, a course on signals and systems may intimidate you. This book was written to prevent that. A good textbook and a good professor are an advantage—but you are the one who does the learning. If you keep the following ideas in mind, you will do very well in this course:

- This course provides a foundation for other courses in the electrical engineering curriculum. For this reason, put in as much effort as you can. Study the course regularly.
- Problem solving is an essential part of the learning process. Solve as many problems as you can. Begin by solving the practice problems following each example, and then proceed to the end-of-chapter problems. The best way to learn is to solve a lot of problems. An asterisk in front of a problem indicates a challenging problem.
- MATLAB® is a software program that is very useful in signals and systems and other courses you will be taking. A brief tutorial on MATLAB is given in Appendix B to get you started. The best way to learn MATLAB is to start with it once you know a few commands.
- Each chapter ends with a section on how the material covered in the chapter can be applied to real-life situations. The concepts in this section may be new and advanced to you. No doubt, you will learn more about the details in other courses. We are mainly interested in gaining a general familiarity with these ideas.
- Attempt the review questions at the end of each chapter. They will help you discover some "tricks" not revealed in class or in the textbook.
- It is very tempting to sell your book after taking this course. However, my advice to you is DO NOT SELL YOUR ENGINEERING BOOKS! When I was a student, I did not sell any of my engineering books and was very glad I did not. I found that I needed most of them throughout my career.

A short review of the mathematical formulas you may need is covered in Appendix A, while answers to odd-numbered problems are given in Appendix C.

Have fun!

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

1 Basic Concepts

Anyone who stops learning is old, whether at twenty or eighty. Anyone who keeps learning stays young. The greatest thing in life is to keep your mind young.

Henry Ford

GLOBAL POSITIONING SYSTEM

The global positioning system (GPS) is a typical illustration of what signals and systems are all about. GPS is a satellite-based navigation system made up of a network of 24 satellites placed into orbit by the U.S. Department of Defense. GPS was originally designed for military use, but in the 1980s, the government made the system available for civilian use.

The 24 satellites that make up the GPS space segment are orbiting the earth about 12,000 miles above us. These satellites travel at speeds of roughly 7,000 miles an hour. GPS satellites transmit two low-power radio signals. The signals travel by line of sight,

DOI: 10.1201/9781003490869-1

meaning they will pass through clouds, glass, and plastic but will not go through most solid objects such as buildings and mountains.

A GPS signal contains three different bits of information—a pseudorandom code, ephemeris data, and almanac data. The pseudorandom code identifies which satellite is transmitting information. You can view this number on your GPS unit's satellite page. Ephemeris data contains important information about the status of the satellite, current date, and time. This part of the signal is essential for determining a position. The almanac data tells the GPS receiver where each GPS satellite should be at any time throughout the day. Each satellite transmits almanac data showing the orbital information for that satellite and for every other satellite in the system.

A GPS receiver calculates its position by precisely timing the signals sent by GPS satellites. The receiver uses the messages it receives to determine the transit time of each message and computes the distance to each satellite. These distances, along with the satellites' locations, are used to compute the position of the receiver.

1.1 INTRODUCTION

The idea of signals and systems arises in different disciplines such as science, engineering, economics, politics, and medicine. Scientists, mathematicians, financial analysts, cardiologists, and engineers all use the concepts of systems and signals because they are the foundation on which we build many things in our daily lives. Typical examples of systems include radio and television, telephone networks, radar systems, computer networks, wireless communication, military surveillance systems, and satellite communication systems. The theory of signals and systems provides a solid foundation for control systems, communication systems, power systems, and networking, to name a few.

Our objective in this book is to present an introductory, yet comprehensive, treatment of signals and systems, with an emphasis on computing using MATLAB®. A knowledge of a broad range of signals and systems is of practical value in describing human experience. It is also important because engineers must be familiar with signal and system concepts and apply the knowledge to analyze specific signals and systems they will deal with in their professional life.

In this chapter, we begin by discussing some basic concepts in signals and systems. We introduce the mathematical representations of signals, their properties, and applications. We also discuss different systems and how the material covered in this chapter is used in some applications. We finally learn how to use MATLAB to process signals.

1.2 BASIC DEFINITIONS

To avoid any misconception, it is expedient that we define at the outset what we mean by signals and systems.

A **signal** $x(t)$ is a set of data or function of time that represents a variable of interest.

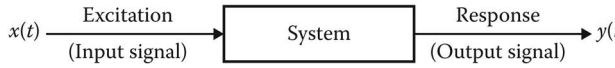


FIGURE 1.1 A simple system.

A signal typically contains information about the nature of a phenomenon. Examples of signals include the atmospheric temperature, humidity, the human voice, television images, a dog's bark, and bird songs. More generally, a signal may be a function of more than one independent variable (time). For example, pictures are signals which depend on two independent variables (horizontal and vertical positions) and may be regarded as a two-dimensional signal. However, in this text, we will consider only one-dimensional signals, with time as the independent variable.

A **system** is a collection of devices that operate on input signal $x(t)$ (or excitation) to produce an output signal $y(t)$ (or response).

A system may also be regarded as a mathematical model of a physical process that relates the input signal to the output signal. In a nutshell, a system transforms input signals into output signals. Examples of systems include electric circuits, computer programs, the stock market, weather, radio receivers, mobile phones, microphones, and the human body. A system may have several mathematical models or representations. The variables in the mathematical model are described as *signals*, which may be current, voltage, or displacement. In electrical systems, signals are often represented as currents and voltages. In mechanical systems, they are often represented as temperatures, forces, and velocities. In hydraulic systems, signals may be displacements and pressures.

Figure 1.1 illustrates the block diagram of a single-input, single-output system. We classify the signals that enter the system as input signals, while the signals produced by the system are classified as outputs. For example, we may regard voltages and currents as functions of time in an electric circuit as signals, while the circuit itself is regarded as a system. In engineering systems, signals may carry energy or information.

1.3 CLASSIFICATION OF SIGNALS

There are many ways of classifying signals: continuous-time or discrete-time, periodic or nonperiodic, energy or power, analog or digital, random or nonrandom, real or complex, etc.

1.3.1 Continuous-Time and Discrete-Time Signals

A signal $x(t)$ that is defined at all instants of time is known as a continuous-time signal.

A **continuous-time signal** takes a value at every instant of time t.

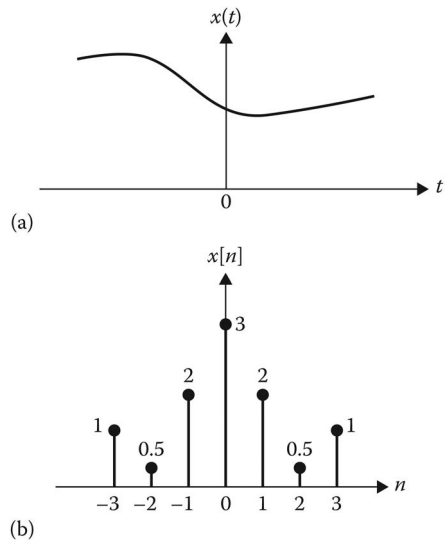


FIGURE 1.2 Typical examples of (a) continuous-time and (b) discrete-time signals.

An example of a continuous-time signal $x(t)$ is shown in Figure 1.2(a).

A **discrete-time signal** is defined only at particular instants of time.

A discrete-time signal is usually identified as a sequence of numbers, denoted by $x[n]$, where n is an integer. It may represent a phenomenon for which the independent variable n is inherently discrete. An example of discrete-time signal $x[n]$ is shown in Figure 1.2(b).

Since time is naturally continuous, most physical systems are continuous-time systems. Discrete-time signals are often obtained from continuous-time signals through sampling.

As shown in Figure 1.3(a), the usual continuous-time signal $x(t)$ is sampled uniformly with sampling period T to produce the discrete-time signal $x[n]$ in Figure 1.3(b). We simplify notation by letting $x(kT) \triangleq x[k]$. A discrete-time signal is equally spaced in time with sampling period T. Thus, discrete-time signals are samples of continuous-time signals, or they may exist naturally.

1.3.2 Periodic and Nonperiodic Signals

A periodic continuous-time signal satisfies

$$x(t) = x(t + nT) \tag{1.1}$$

where n is an integer and T is the period of the signal.

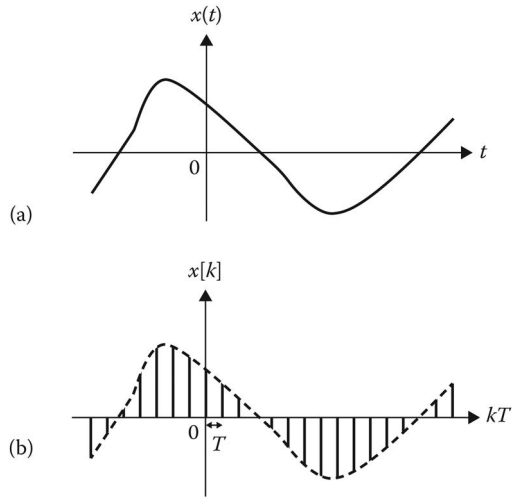


FIGURE 1.3 Obtaining $x[k]$ from $x(t)$ through sampling.

A **periodic signal** is the one that repeats itself every T seconds.

A popular example of a periodic signal is the sinusoid

$$x(t) = A\sin(\omega t + \theta), \quad -\infty < t < \infty \tag{1.2}$$

where A is the amplitude of the signal, ω ($= 2\pi f = 2\pi/T$) is the angular frequency in radians per second, and θ is the phase in radians. Another example of a periodic (nonsinusoidal) continuous-time signal is shown in Figure 1.4(a). Any signal that does not satisfy the periodicity condition in eq. (1.1) is called a *nonperiodic* signal. We will be dealing with nonperiodic signals (step function, ramp functions, rectangular functions, etc.) later.

A discrete-time signal $x[n]$ is periodic with period N if it satisfies

$$x[n] = x[n+N] \tag{1.3}$$

The discrete-time sine and cosine signals are given in terms of complex exponential signals as

$$x[n] = \cos(\omega n) = \frac{1}{2}(e^{j\omega n} + e^{-j\omega n}) \tag{1.4}$$

$$y[n] = \sin(\omega n) = \frac{1}{2j}(e^{j\omega n} - e^{-j\omega n}) \tag{1.5}$$

Another example of a periodic discrete-time signal is shown in Figure 1.4(b).

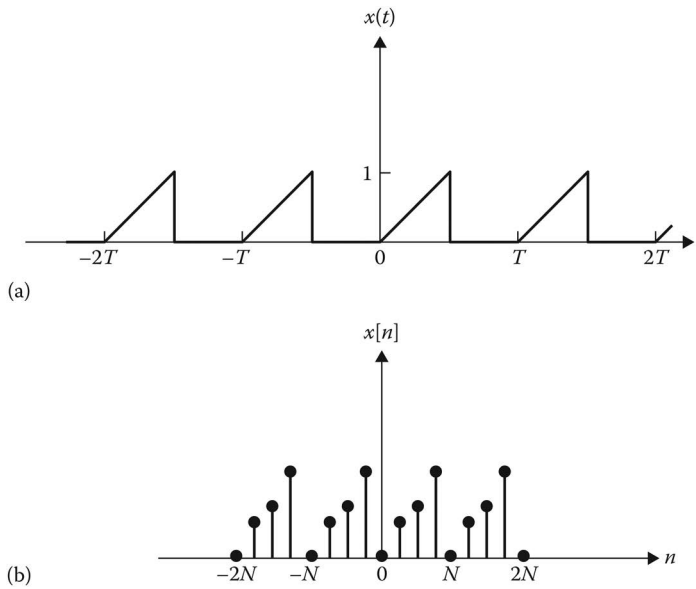


FIGURE 1.4 Examples of periodic (a) continuous-time and (b) discrete-time signals.

1.3.3 Analog and Digital Signals

If a continuous-time signal $x(t)$ can assume any value in the range $-\infty < t < \infty$, then it is called an analog signal. Although all analog signals are continuous-time signals, not all continuous-time signals are analog signals. If a discrete-time signal assumes only finite values, then it is called a digital signal.

An **analog signal** is a continuous-time signal in which the variation with time is analogous (or proportional) to some physical phenomenon.

A **digital signal** is a discrete-time signal that can have a finite number of values (usually binary).

A digital signal can assume only a finite number of values. The difference between analog and digital signals is that analog is a continuous electrical signal, whereas digital is a discrete electrical signal. We live in an analog world, and most signals are analog. Although some signals are inherently digital, most digital signals are obtained from analog signals by sampling or an analog-to-digital converter (ADC).

For example, an analog signal is taken straight from the microphone and recorded into tape in its original form. The signal from the microphone is an analog signal, and therefore the signal on the tape is analog as well. Since data is sent using variable currents in an analog system, it is very difficult to remove noise and signal

distortions during the transmission. For this reason, analog signals cannot perform high-quality data transmission. On the other hand, digital signals use binary data strings (0 and 1) to reproduce data being transmitted.

1.3.4 Energy and Power Signals

For continuous-time signal $x(t)$, the normalized energy E of $x(t)$ [assuming $x(t)$ is real] is

$$E = \int_{-\infty}^{\infty} x(t)^2 \, dt \tag{1.6}$$

If $x(t)$ is complex valued, eq. (1.6) can be generalized to

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{1.7}$$

where $|x(t)|$ is the magnitude of $x(t)$. The normalized power P for real $x(t)$ is

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)^2 \, dt \tag{1.8}$$

This can be generalized for complex value $x(t)$ as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \tag{1.9}$$

Similarly, for a discrete-time signal $x[n]$, the normalized energy E of $x[n]$ is

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{1.10}$$

while the normalized power P is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{1.11}$$

Based on the definitions of E and P in eqs. (1.6)–(1.11), we define the following.

A signal $x(t)$ or $x[n]$ is an **energy signal** if and only if $0 < E < \infty$ and consequently $P = 0$.

A signal $x(t)$ or $x[n]$ is a **power signal** if and only if $0 < P < \infty$ and consequently $E = \infty$.

If a signal is a power signal, then it cannot be an energy signal or vice versa; power and energy signals are mutually exclusive. A signal may be neither power nor energy signal if the conditions in eqs. (1.6)–(1.11) are not met. Almost all periodic functions of practical interest are power signals.

1.3.5 Even and Odd Symmetry

By definition, a signal is even if

$$x(t) = x(-t) \tag{1.12}$$

A function is even if its plot is symmetrical about the vertical axis; i.e., the signal for $t < 0$ is the mirror image of the signal for $t > 0$. Examples of even signals are cos t, t^2, and t^4.

By definition, a signal is odd if

$$x(t) = -x(-t) \tag{1.13}$$

The plot of an odd function is antisymmetrical about the vertical axis. Examples of odd functions are t, t^3, and sin t.

An **even signal** $x(t)$ is one for which $x(t) = x(-t)$ and an **odd signal** $y(t)$ is one for which $y(t) = -y(-t)$.

Any signal $x(t)$ can be represented as the sum of even and odd signals as

$$x(t) = x_e(t) + x_o(t) \tag{1.14}$$

where $x_e(t)$ is the even part and $x_o(t)$ is the odd part. Replacing t with $-t$ in eq. (1.14) and incorporating eqs. (1.12) and (1.13), we get

$$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t) \tag{1.15}$$

Adding eqs. (1.14) and (1.15) and dividing by 2,

$$x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] \tag{1.16}$$

Subtracting eq. (1.15) from eq. (1.14) and dividing by 2,

$$x_o(t) = \frac{1}{2}\big[x(t) - x(-t)\big] \tag{1.17}$$

Thus,

$$x(t) = \underbrace{\frac{x(t) + x(-t)}{2}}_{even} + \underbrace{\frac{x(t) - x(-t)}{2}}_{odd} \tag{1.18}$$

Equation (1.16) shows that the two signals are added and scaled in magnitude to produce the even signal $x_e(t)$, while eq. (1.17) indicates that $x(–t)$ is subtracted from $x(t)$ and the result is amplitude-scaled by 0.5 to produce the odd signal $x_o(t)$. For a discrete-time signal $x[n]$, we can construct the even and odd parts using eqs. (1.16) and (1.17).

Note the following properties of even and odd functions:

1. The product of two even functions is also an even function.
2. The product of two odd functions is an even function.
3. The product of an even function and an odd function is an odd function.
4. The sum (or difference) of two even functions is also an even function.
5. The sum (or difference) of two odd functions is an odd function.
6. The sum (or difference) of an even function and an odd function is neither even nor odd.

Each of these properties can be proved using eqs. (1.12) and (1.13).

Example 1.1

Show that the signal in eq. (1.2) is periodic.

SOLUTION

The period of the signal in eq. (1.2) is

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\begin{aligned} x(t+T) &= A\sin[\omega(t+T)+\theta] = A\sin\left[\omega\left(t+\frac{2\pi}{\omega}\right)+\theta\right] \\ &= A\sin(\omega t + 2\pi + \theta) = A\sin(\omega t + \theta) \\ &= x(t) \end{aligned}$$

showing that $x(t)$ satisfies eq. (1.1); hence, it is periodic.

Practice Problem 1.1 Show that the signal $x(t) = A\cos(2\pi t + 0.1\pi)$ is periodic.

Answer: Proof.

Example 1.2

Determine whether the following signals are energy signals, power signals, or neither.

a. $x(t) = \begin{cases} e^{-at}, & 0 < t < \infty, a > 0 \\ 0, & \text{otherwise} \end{cases}$

b. $x(t) = A\cos(\omega t + \theta)$

c. $x[n] = 10e^{j2n}$

SOLUTION

a. The normalized energy of the signal is

$$E = \int_{-\infty}^{\infty} x(t)^2\,dt = \int_{0}^{\infty} e^{-2at}\,dt = \frac{e^{-2at}}{-2a}\bigg|_{0}^{\infty} = \frac{1}{2a} < \infty$$

confirming that $x(t)$ is an energy signal.

b. The sinusoidal signal has period $T = 2\pi/\omega$.
The normalized average power is

$$p = \frac{1}{T}\int_{0}^{T} x(t)^2\,dt = \frac{\omega}{2\pi}\int_{0}^{T} A^2\cos^2(\omega t + \theta)\,dt$$

$$= \frac{A^2\omega}{2\pi}\int_{0}^{2\pi/\omega} \frac{1}{2}\left[1 + \cos(2\omega t + 2\theta)\right]dt$$

$$= \frac{A^2\omega}{2\pi}\,\frac{1}{2}\,\frac{2\pi}{\omega} = \frac{A^2}{2} < \infty$$

showing that $x(t)$ is a power signal. All periodic signals are generally power signals.

c. $|x[n]| = |10\,e^{j2n}| = 10|e^{j2n}| = 10\,|\cos(2n) + j\sin(2n)| = 10\sqrt{[\cos^2(2n) + \sin^2(2n)]} = 10$

The normalized average power is

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x[n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}10^2$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}100(2N+1) = 100 < \infty$$

i.e., $x[n]$ is a power signal.

Practice Problem 1.2 Determine whether the following signals are energy signals, power signals, or neither.

a. $x(t) = \begin{cases} t, & 0 < t < \infty \\ 0, & \text{otherwise} \end{cases}$

b. $x(t) = \begin{cases} A, & -a < t < a \\ 0, & \text{otherwise} \end{cases}$

c. $x[n] = 5e^{-j4n}$

Answers: (a) neither, (b) energy signal, (c) power signal.

Example 1.3

Determine the even and odd components of:

a. $x(t) = 10\sin t + 5\cos t - 2\cos t \sin t$
b. $y(t)$ shown in Figure 1.5(a)

SOLUTION

a. We first find $x(-t)$ and then apply eqs. (1.13) and (1.14). We keep in mind that $\sin(-t) = -\sin t$ and $\cos(-t) = \cos t$.

$$\begin{aligned} x(-t) &= 10\sin(-t) + 5\cos(-t) - 2\cos(-t)\sin(-t) \\ &= -10\sin t + 5\cos t + 2\cos t \sin t \end{aligned}$$

Hence,

$$x_e(t) = \frac{x(t) + x(-t)}{2} = 5\cos t$$
$$x_o(t) = \frac{x(t) - x(-t)}{2} = 10\sin t - 2\cos t \sin t$$

b. We first obtain $y(-t)$ as shown in Figure 1.5(b). Note that

$$y(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$
$$y(-t) = \begin{cases} 1, & t < 0 \\ 0, & t > 0 \end{cases}$$

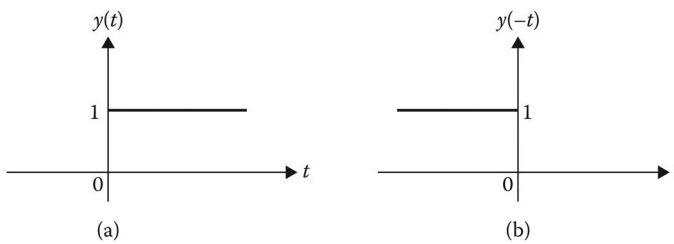


FIGURE 1.5 For Example 1.3(b).

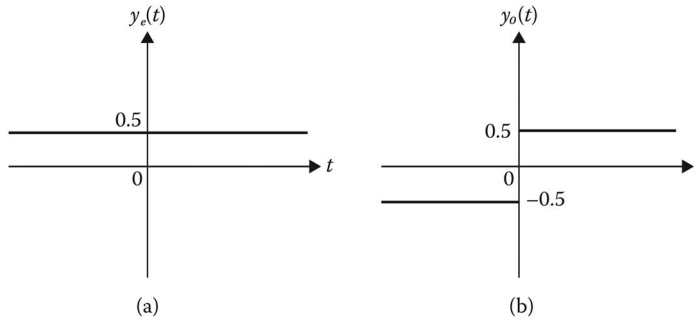


FIGURE 1.6 For Example 1.3(b).

Hence,

$$y_e(t) = \frac{y(t)+y(-t)}{2} = \begin{cases} 0.5(0+1), & t<0 \\ 0.5(1+0), & t>0 \end{cases} = 0.5$$

$$y_o(t) = \frac{y(t)-y(-t)}{2} = \begin{cases} 0.5(0-1), & t<0 \\ 0.5(1-0), & t>0 \end{cases} = \begin{cases} -0.5, & t<0 \\ 0.5, & t>0 \end{cases}$$

Both $y_e(t)$ and $y_o(t)$ are sketched in Figure 1.6.

Practice Problem 1.3 Find the even and odd parts of the following signals:

a. $z(t) = t^2 + 4t - 10$
b. $h(t)$ shown in Figure 1.7

Answers:

a. $z_e(t) = t^2 - 10$, $z_o(t) = 4t$;

b. $h_e(t) = \frac{1}{2}[u(t+1) - u(t-1)]$,

$h_o(t) = \frac{1}{2}[-u(t+1) + 2u(t) - u(t-1)]$; see Figure 1.8.

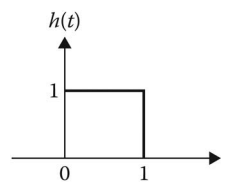


FIGURE 1.7 For Practice Problem 1.3(b).

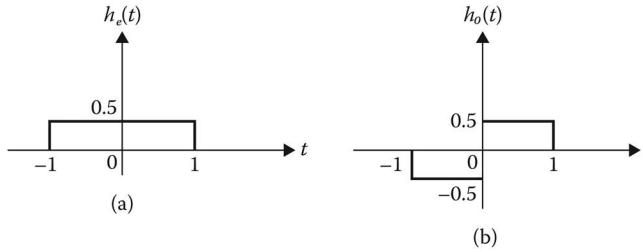


FIGURE 1.8 For Practice Problem 1.3(b).

1.4 BASIC CONTINOUS-TIME SIGNALS

We now consider some simple, standard continuous-time signals. These include the unit step function u(t), the unit ramp function $r(t)$, and the impulse function $\delta(t)$. These three functions are called singularity functions. A singularity function is one that is discontinuous or has discontinuous derivatives. In addition to the singularity functions, we will also consider the unit rectangular function $\Pi(t)$, the unit triangular function $\Lambda(t)$, the sinusoidal function, and the exponential function.

1.4.1 Unit Step Function

The *unit step function u(t)*, also known as the *Heaviside unit function*, is defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.19}$$

The **unit step function** $u(t)$ is 0 for negative values of t and 1 for positive values of t.

The unit step function is shown in Figure 1.9. We should note that $u(t)$ is discontinuous at $t = 0$; that is, it is undefined at $t = 0$.

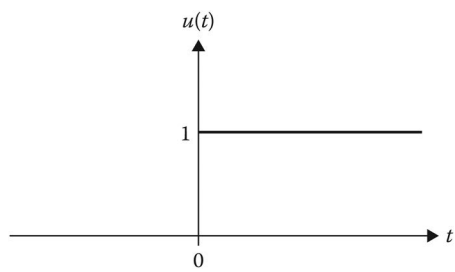


FIGURE 1.9 The unit step function.

We use the step function to represent an abrupt change, like the changes that occur in the circuits of control systems and digital computers. It is used in representing a signal that is zero up to some time and finite thereafter. For example, the signal

$$x(t) = \begin{cases} \cos \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.20}$$

can be written concisely as

$$x(t) = u(t) \cos \omega t \tag{1.21}$$

As another example, when we turn the key in the ignition system of a car, we are actually introducing a step voltage, the battery voltage. Given a continuous-time signal $x(t)$, the product $x(t)u(t)$ is given by

$$x(t)u(t) = \begin{cases} x(t), & t > 0 \\ 0, & t < 0 \end{cases} \tag{1.22}$$

This is illustrated in Figure 1.10. From this, we conclude that the multiplication of a signal $x(t)$ with the unit step function $u(t)$ eliminates the values of $x(t)$ for $t < 0$.

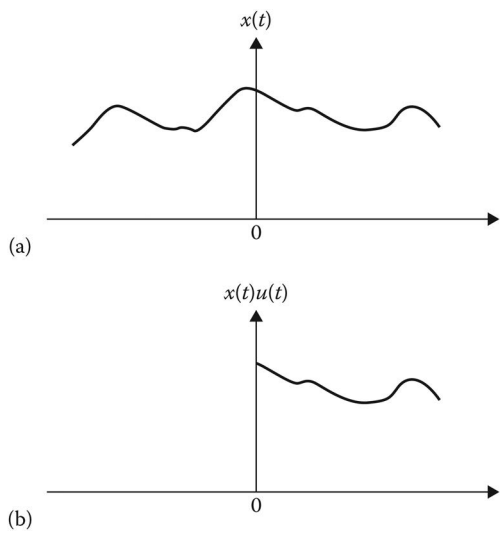


FIGURE 1.10 An arbitrary signal $x(t)$ in (a) is multiplied by $u(t)$ in (b).

1.4.2 Unit Impulse Function

The derivative of the unit step function $u(t)$ is the *unit impulse function* $\delta(t)$, which we write as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases} \tag{1.23a}$$

or

$$\int_{-\infty}^{t} \delta(t)\,dt = u(t) \tag{1.23b}$$

The unit impulse function, also known as the *delta function*, is shown in Figure 1.11.

The **unit impulse function** $\delta(t)$ is zero everywhere, except at $t = 0$, where it is undefined.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized as a very short-duration pulse of unit area. This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)\,dt = 1 \tag{1.24}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as shown in Figure 1.12. For example, an impulse function $10\delta(t)$ has an area of 10. Figure 1.12 shows the impulse functions $5\delta(t + 2)$, $10\delta(t)$, and $-4\delta(t - 3)$.

To better understand the delta function, consider the function shown in Figure 1.13(a). The function $g(t)$ becomes a unit step as $\Delta \to 0$.

$$u(t) = \lim_{\Delta \to 0} g(t) \tag{1.25}$$

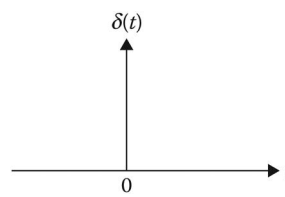


FIGURE 1.11 The unit impulse function.

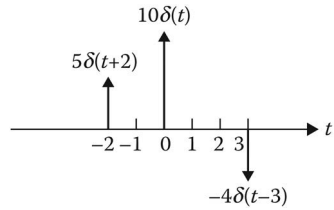


FIGURE 1.12 Three impulse functions.

The derivative $g'(t)$ of $g(t)$ is shown in Figure 1.13(b), where we notice that as $\Delta \to 0$, $g'(t)$ becomes the unit impulse function.

$$\delta(t) = \lim_{\Delta \to 0} g'(t) \tag{1.26}$$

Thus, we can consider the delta function $\delta(t)$ as a large spike or impulse at the origin.

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_o)\, dt \tag{1.27}$$

where $a < t_o < b$. Since $\delta(t - t_o) = 0$ except at $t = t_o$, the integrand is zero except at t_o.

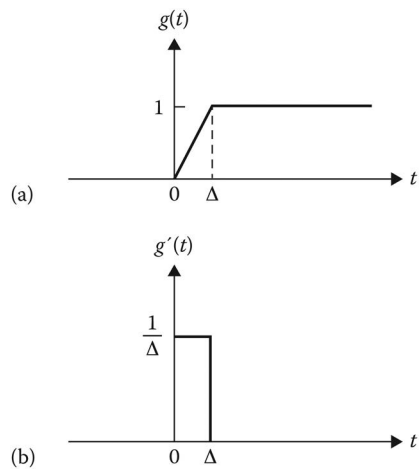


FIGURE 1.13 Functions that approach the unit step and unit impulse as $\Delta \to 0$.

TABLE 1.1
Properties of the Unit Impulse Function

1. $f(t)\delta(t-t_o) = f(t_o)\delta(t-t_o)$
2. $\int_{-\infty}^{\infty} f(t)\delta(t-t_o)\,dt = f(t_o)$
3. $\delta(t-t_o) = \frac{d}{dt}u(t-t_o)$
4. $u(t-t_o) = \int_{-\infty}^{t} \delta(\lambda-t_o)\,d\lambda = \begin{cases} 1, & t > t_o \\ 0, & t < t_o \end{cases}$
5. $\delta(-t) = \delta(t)$, i.e., $\delta(t)$ is an even function

Thus,

$$\int_a^b f(t)\delta(t-t_o)\,dt = \int_a^b f(t_o)\delta(t-t_o)\,dt = f(t_o)\int_a^b \delta(t-t_o)\,dt = f(t_o)$$

or

$$\int_a^b f(t)\delta(t-t_o)\,dt = f(t_o) \tag{1.28}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a useful property of the impulse function known as the *sampling or sifting* property. The special case of eq. (1.28) is for $t_o = 0$. Then, eq. (1.28) becomes

$$\int_{0^-}^{0^+} f(t)\delta(t)\,dt = f(0) \tag{1.29}$$

If the integration interval does not include the impulse function, then the integration in eq. (1.23) is zero. For example,

$$\int_{-\infty}^{-2} \delta(t)\,dt = 0, \qquad \int_{3}^{5} \delta(t)\,dt = 0, \tag{1.30}$$

This and other properties of the impulse functions are presented in Table 1.1.

1.4.3 Unit Ramp Function

Integrating the unit step function $u(t)$ results in the *unit ramp function* $r(t)$; we write

$$r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda = tu(t) \tag{1.31}$$

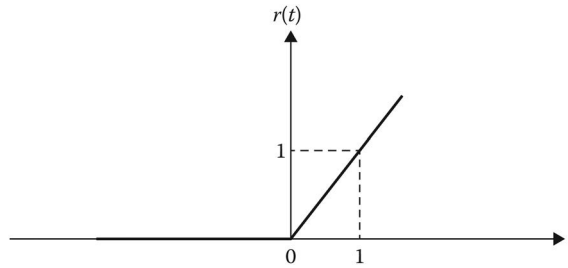


FIGURE 1.14 The unit ramp function.

or

$$r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases} \tag{1.32}$$

The **unit ramp function** is zero for negative values of t and has a unit slope for positive values of t.

Figure 1.14 shows the unit ramp function. In general, a ramp is a function that changes at a constant rate.

We should keep in mind that the three singularity functions (step, impulse, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \quad u(t) = \frac{dr(t)}{dt} \tag{1.33}$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda, \quad r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda \tag{1.34}$$

1.4.4 Rectangular Pulse Function

The unit rectangular pulse function is defined as

$$\Pi(t/\tau) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -\tau/2 < t < \tau/2 \\ 0, & \text{otherwise} \end{cases} \tag{1.35}$$

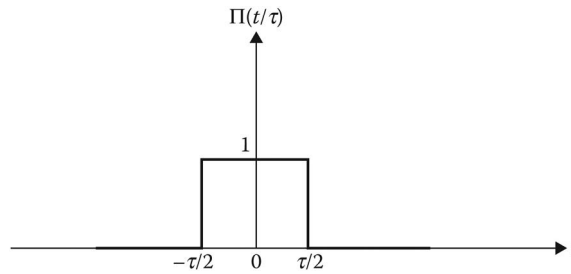


FIGURE 1.15 The unit rectangular pulse function.

It is centered at the origin and has unit height and width τ. Figure 1.15 shows the rectangular pulse function. It is evident from the figure that we can express the rectangular pulse function in terms of the unit step function as

$$\Pi(t/\tau) = u(t+\tau/2) - u(t-\tau/2) \tag{1.36}$$

The rectangular pulse function results from an on-off switching operation of a source. It is used in extracting part of a signal.

1.4.5 Triangular Pulse Function

The unit triangular pulse function is defined as

$$\Lambda(t/\tau) = \begin{cases} 1-\dfrac{|t|}{\tau}, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1+\dfrac{t}{\tau}, & -\tau < t < 0 \\ 1-\dfrac{t}{\tau}, & 0 < t < \tau \\ 0, & \text{otherwise} \end{cases} \tag{1.37}$$

This means that the unit triangular pulse is centered at the origin and has unit height and width 2τ. The triangular pulse function is shown in Figure 1.16.

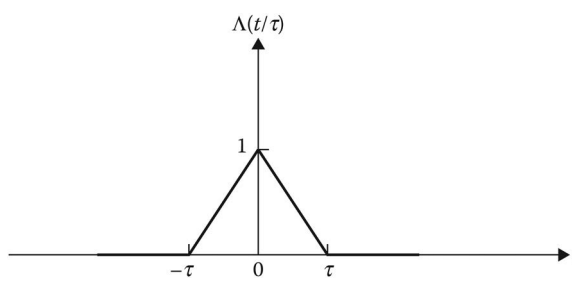


FIGURE 1.16 The unit triangular pulse function.

1.4.6 Sinusoidal Signal

Perhaps the most important and useful of all the standard signals is the sine wave, or *sinusoidal signal*. It can take the form of

$$x(t) = \sin \omega t, \quad x(t) = \cos \omega t \tag{1.38}$$

or by combining both to form a complex exponential signal

$$x(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t \tag{1.39}$$

where ω is the frequency of the sinusoid. We already came across sinusoids in eq. (1.2).

1.4.7 Exponential Signal

The continuous-time exponential function is defined as

$$x(t) = Ae^{at} \tag{1.40}$$

where A and a are constants, which may be complex in general. If we assume that A and a are real, the exponential function is shown in Figure 1.17 for both positive and negative values of a. The signal $x(t)$ in eq. (1.39) is a complex exponential signal, which will play a major role in our treatment of signals and systems. One reason that causes the signal to arise in many applications is its rate of change.

$$\frac{d}{dt}x(t) = aAe^{at} = ax(t) \tag{1.41}$$

This means that the derivative of $x(t)$ at any time is proportional to its value at that time.

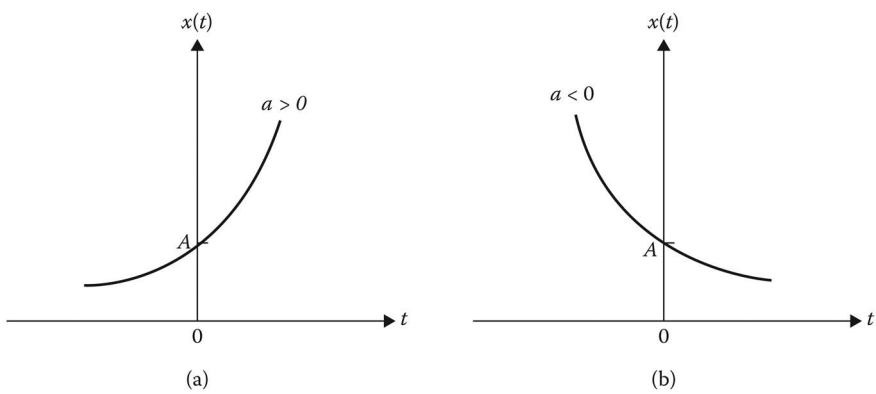


FIGURE 1.17 The exponential signal: (a) $a > 0$ and (b) $a < 0$.

Example 1.4

Evaluate the following:

a. $\int_{-\infty}^{\infty} \sin 3t\,\delta\left(t-\frac{\pi}{2}\right)$

b. $\int_{-\infty}^{\infty} t^4\delta(t-2)\,dt$

SOLUTION

a. $$\int_{-\infty}^{\infty} \sin 3t\,\delta\left(t-\frac{\pi}{2}\right) = \sin 3t\Big|_{t=\pi/2}\,\delta\left(t-\frac{\pi}{2}\right)$$
$$= \sin\frac{3\pi}{2}\,\delta\left(t-\frac{\pi}{2}\right) = -\delta\left(t-\frac{\pi}{2}\right)$$

b. $$\int_{-\infty}^{\infty} t^4\delta(t-2)\,dt = t^4\Big|_{t=2} = 2^4 = 16$$

Practice Problem 1.4 Evaluate:

a. $\int_{-\infty}^{\infty}\left[\sin\left(t^3+\frac{\pi}{2}\right)\right]\delta(t)\,dt$

b. $\int_{0}^{10}(t^2+4t-2)\delta(t-1)\,dt$

Answer: (a) 1, (b) 3

Example 1.5

Express the pulse (or rectangular) in Figure 1.18 in terms of the unit step. Calculate its derivative and sketch it.

SOLUTION

The signal in Figure 1.18 is also called the *gate function*. It may be regarded as a step function that switches on at $t = 2$ seconds and switches off at $t = 5$ seconds.

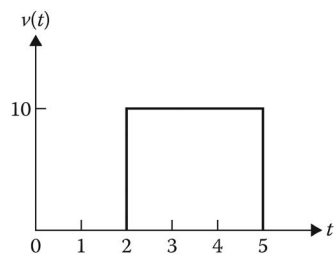


FIGURE 1.18 For Example 1.4.

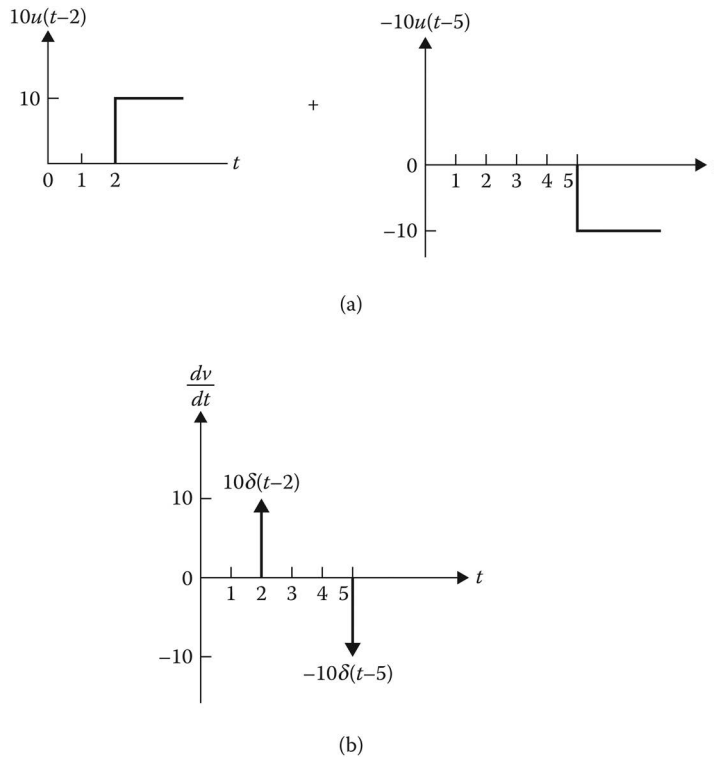


FIGURE 1.19 For Example 1.5: (a) decomposition of the pulse in Figure 1.18 and (b) derivative of the pulse in Figure 1.18.

As shown in Figure 1.19(a), it consists of two step functions. From the figure, we get

$$v(t) = 10u(t-2) - 10u(t-5) = 10\left[u(t-2) - u(t-5)\right]$$

Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Figure 1.19(b).

Practice Problem 1.5 (a) Express the current pulse in Figure 1.20 in terms of the unit step function. (b) Determine its derivative and sketch it. (c) Find its integral and sketch it.

Answer: (a) $(10[u(t) - 2u(t-2) + u(t-4)]$. (b) $10\delta(T) - 20\delta(T-2) + 10\delta(t-4)$; see Figure 1.21(a). (c) $10[r(t) - 2r(t-2) + r(t-4)]$; see Figure 1.21(b).

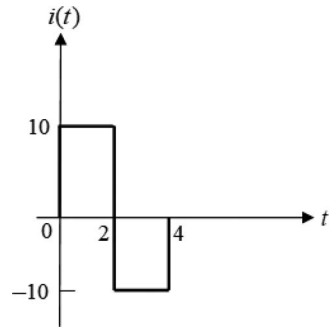


FIGURE 1.20 For Practice Problem 1.5.

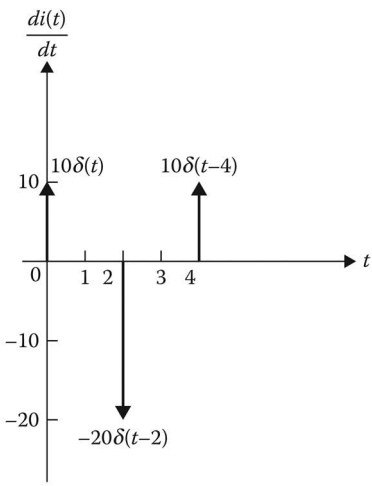
Example 1.6

Consider the signal $y(t)$ in Figure 1.22. Find its first derivative $y'(t)$ and its second derivative $y''(t)$.

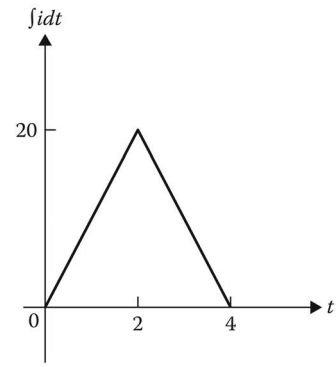
SOLUTION

From Figure 1.22, we can write $y(t)$ as

$$y(t) = \begin{cases} 4t, & 0 < t < 1 \\ -2t + 6, & 1 < t < 3 \end{cases}$$



(a)



(b)

FIGURE 1.21 For Practice Problem 1.5.

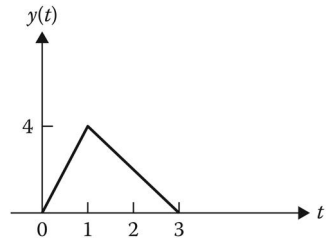


FIGURE 1.22 For Example 1.6.

If we take the first derivative, we obtain

$$y'(t) = \begin{cases} 4, & 0 < t < 1 \\ -2, & 1 < t < 3 \end{cases}$$

which is sketched in Figure 1.23(a). From the previous example, we notice that $y'(t)$ consists of two gate functions. Taking its derivative produces three abrupt changes, which result in three impulse functions as shown in Figure 1.23(b), i.e.,

$$y''(t) = 4\delta(t) - 6\delta(t-1) + 2\delta(t-3)$$

Practice Problem 1.6 Find and sketch the first and second derivatives of the signal $y(t)$ in Figure 1.24.

Answer: $y'(t) = \begin{cases} 2, & 0 < t < 1 \\ 0, & 1 < t < 3 \\ -2, & 3 < t < 4 \end{cases}$, $y''(t) = 2\delta(t) - 2\delta(t-1) - 2\delta(t-3) + 2\delta(t-4)$

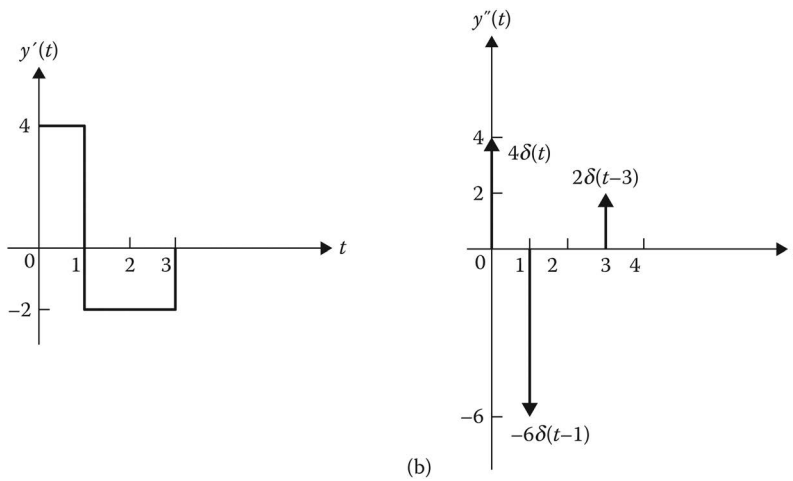


FIGURE 1.23 For Practice Problem 1.6.

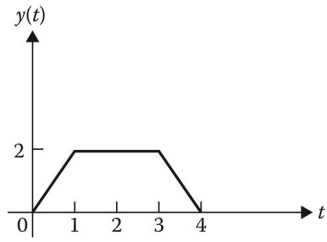


FIGURE 1.24 The unit step sequence.

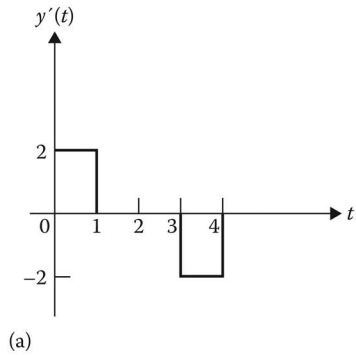
They are sketched in Figure 1.25.

Example 1.7

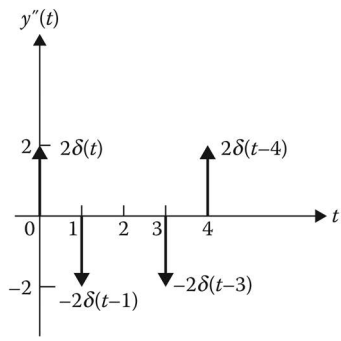
Sketch the following signals:

a. $x(t) = 3\Pi\left(\frac{t-4}{2}\right)$

b. $y(t) = 2\Lambda\left(\frac{t-3}{2}\right)$



(a)



(b)

FIGURE 1.25 For Practice Problem 1.6.

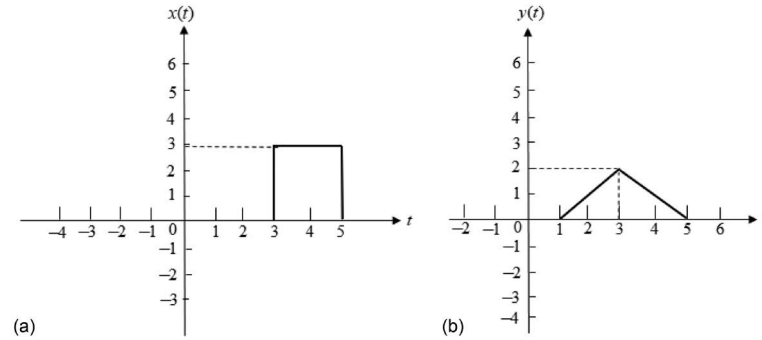


FIGURE 1.26 For Example 1.7.

SOLUTION

a. The signal $x(t)$ is a rectangular pulse centered at $t = 4$ with magnitude 3 and has width 2. It is sketched in Figure 1.26(a). To be sure that the sketch is correct, we use eq. (1.35). Replace every t with $t - 4$ and set $\tau = 2$. We obtain

$$x(t) = 3\Pi\left(\frac{t-4}{2}\right) = 3\begin{cases} 1, & -2/2 < t-4 < 2/2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 3, & -1 < t-4 < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 3, & 3 < t < 5 \\ 0, & \text{otherwise} \end{cases}$$

which is exactly what we have in Figure 1.26(a).

b. This is a triangular pulse centered at $t = 3$ with magnitude 2 and with $\tau = 2$ so that the width is $2\tau = 4$. Thus, $y(t)$ is sketched in Figure 1.26(b). To confirm this, we use eq. (1.37). Replace every t with $t - 3$ and set $\tau = 2$. We obtain

$$y(t) = 2\Lambda\left(\frac{t-3}{2}\right) = 2\begin{cases} 1+\dfrac{t-3}{2}, & -2 < t-3 < 0 \\ 1-\dfrac{t-3}{2}, & 0 < t-3 < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2+t-3, & 1 < t < 3 \\ 2-t+3, & 3 < t < 5 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} t-1, & 1 < t < 3 \\ 5-t, & 3 < t < 5 \\ 0, & \text{otherwise} \end{cases}$$

which confirms the sketch in Figure 1.26(b).

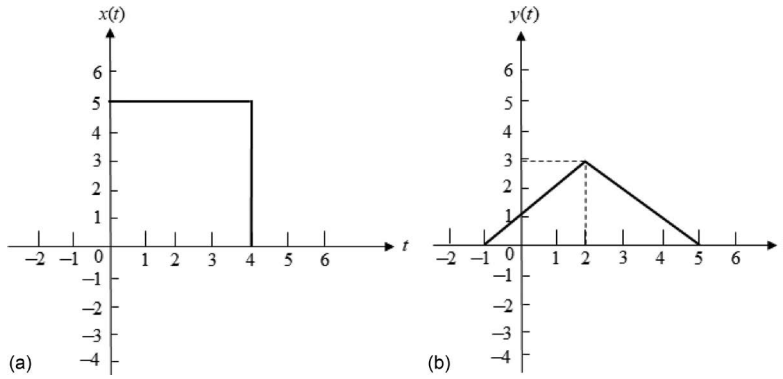


FIGURE 1.27 For Practice Problem 1.7.

Practice Problem 1.7 Sketch the following signals:

a. $x(t) = 5\Pi\left(\frac{t-2}{4}\right)$

b. $y(t) = 3\Lambda\left(\frac{t-2}{3}\right)$

Answer: See Figure 1.27.

1.5 BASIC DISCRETE-TIME SIGNALS

Here we consider some simple, standard discrete-time signals.

1.5.1 The Unit Step Sequence

Like the continuous-time unit step, we define the unit step sequence $u[n]$ as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{1.42}$$

As shown in Figure 1.28, $u[n]$ is a sequence of 1 second starting at the origin. Notice that $u[n]$ is defined at $n = 0$, unlike $u(t)$, which is not defined at $t = 0$.

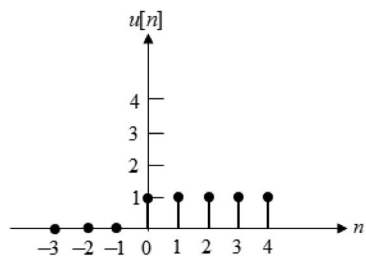


FIGURE 1.28 The unit step sequence.

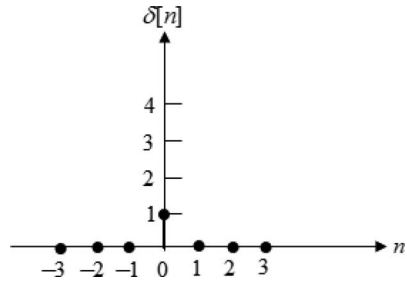


FIGURE 1.29 The unit impulse sequence.

1.5.2 The Unit Impulse Sequence

In discrete time, we define the unit impulse sequence as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{1.43}$$

This is illustrated in Figure 1.29. Notice that we do not have difficulties in defining $\delta[n]$, unlike $\delta(t)$. Some properties of the unit impulse sequence are listed in Table 1.2.

1.5.3 Unit Ramp Sequence

The unit ramp sequence is defined as

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{1.44}$$

The sequence is shown in Figure 1.30. The relationships between unit impulse, unit step, and unit ramp sequences are:

$$\delta[n] = u[n] - u[n-1] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.45a}$$

TABLE 1.2
Properties of the Unit Impulse Sequence

1. $x[n]\delta[n] = x[0]\delta[n]$
2. $x[n]\delta[n-k] = x[k]\delta[n-k]$
3. $\delta[n] = u[n] - u[n-1]$
4. $u[n] = \sum_{k=-\infty}^{n} \delta[k]$
5. $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

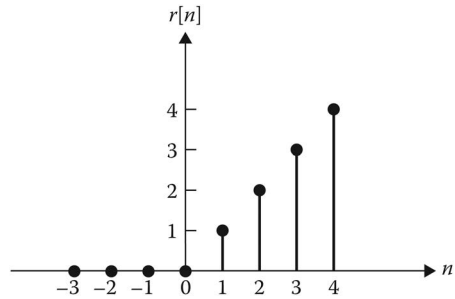


FIGURE 1.30 The unit ramp sequence.

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \tag{1.45b}$$

$$u[n] = r[n+1] - r[n] \tag{1.45c}$$

$$r[n] = \sum_{m=-\infty}^{n-1} u[m] \tag{1.45d}$$

1.5.4 Sinusoidal Sequence

The sinusoidal sequence, or a discrete-time sinusoid, is given by

$$x[n] = A\cos\left(\frac{2\pi n}{N} + \theta\right) = \text{Re}\left[Ae^{j\left(\frac{2\pi n}{N}+\theta\right)}\right] \tag{1.46}$$

where A is a positive real number and is the amplitude of the sequence, N is the period, θ is the phase, and n is an integer. A typical sinusoidal sequence for $A = 1$, $N = 12$, and $\theta = 0$ is shown in Figure 1.31.

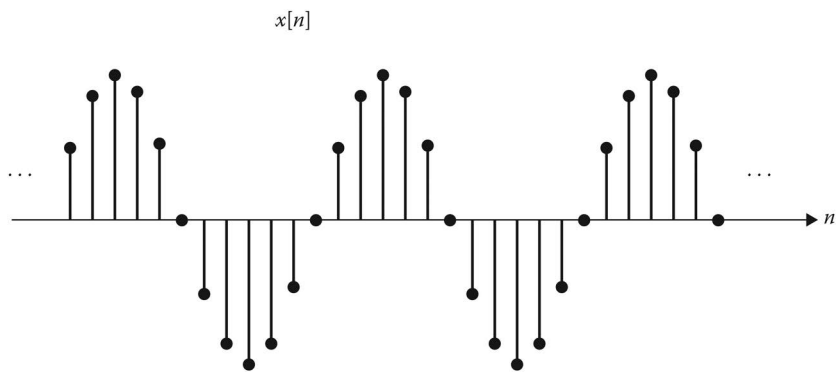


FIGURE 1.31 A discrete-time sinusoidal sequence, $x[n] = \cos(\pi n/6)$.

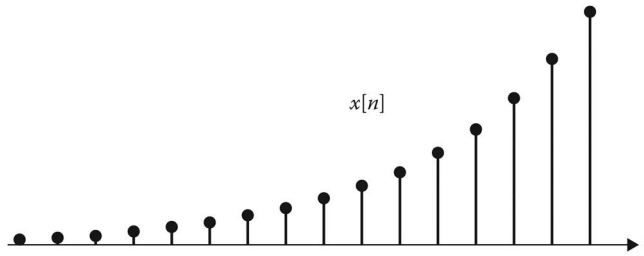


FIGURE 1.32 A discrete-time exponential sequence, $\alpha > 1$.

1.5.5 Exponential Sequence

If we sample a continuous-time exponential function $x(t) = Ae^{-at}$ with sampling period T, we obtain the sequence $x[n] = Ae^{-anT} = A\alpha^n$, with $\alpha = e^{-aT}$. Thus the exponential sequence is given by

$$x[n] = A\alpha^n \tag{1.47}$$

where A and α are generally complex numbers and n is an integer. A typical discrete-time exponential sequence is shown in Figure 1.32. For the signal shown in Figure 1.32, both A and α are real numbers.

Example 1.8

If $r[n] = nu[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$, find $y[n] = 2\,r[1 - n]$.

SOLUTION

To obtain $y[n]$, we replace every n in $r[n]$ with $-n + 1$.

$$y[n] = 2r[-n+1] = 2(-n+1)u[-n+1] = \begin{cases} -2n+2, & -n+1 \geq 0 \\ 0, & -n+1 < 0 \end{cases}$$

$$= \begin{cases} -2n+2, & n \leq 1 \\ 0, & n > 1 \end{cases}$$

Practice Problem 1.8 Given $r[n]$ in Example 1.8, obtain $z[n] = r(n + 2)$.

Answer: $z[n] = \begin{cases} n+2, & n \geq -2 \\ 0, & n < -2 \end{cases}$

Example 1.9

Write down expressions for the sequences shown in Figure 1.33.

SOLUTION

We use item 5 in Table 1.2 as a general way of expressing any discrete signal:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

a. For the discrete-time signal in Figure 1.33(a),

$$x[n] = \delta(n) + 2\delta(n-3)$$

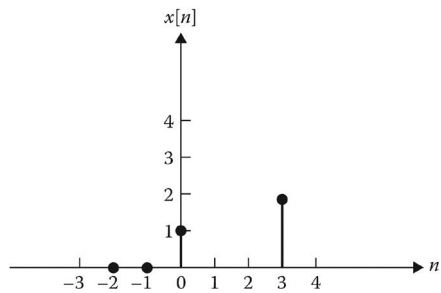
b. Similarly, for $y[n]$ in Figure 1.33(b),

$$y[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 2\delta[n-2]$$

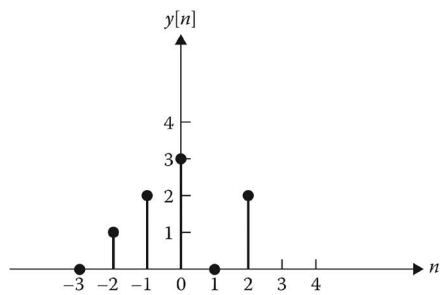
Practice Problem 1.9 Write down expressions for the sequences shown in Figure 1.34.

Answers:

a. $x[n] = \delta[n + 1] + 2\delta[n - 1]$
b. $y[n] = 2\delta[n + 2] + \delta[n + 1] + \delta[n - 2] + 2\delta[n - 2]$



(a)



(b)

FIGURE 1.33 For Example 1.9.

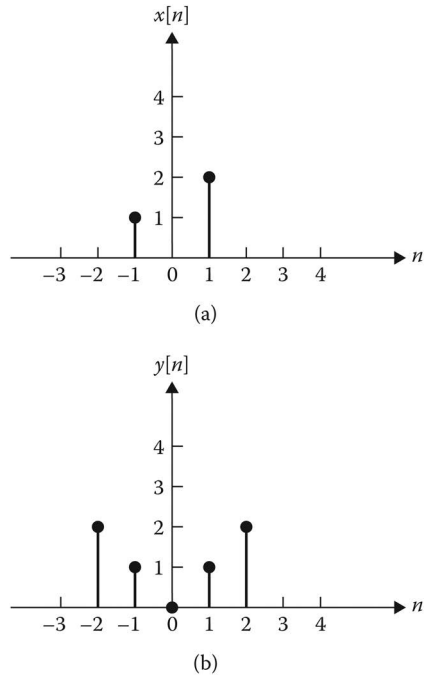


FIGURE 1.34 For Practice Problem 1.9.

1.6 BASIC OPERATIONS ON CONTINOUS SIGNALS

We have limited ourselves to considering signals with one independent variable, time (t). We consider six basic operations or transformations on real functions. The first three operations have to do with time, while the second three deal with transformations in amplitude. The combinations of these transformations make it possible to obtain complex signals from the basic, standard signals. Signals that are produced by these transformations are useful in sonar, radar, signal processing, and communication systems.

1.6.1 Time Reversal

Given a signal $x(t)$, its time reversal is $x(–t)$. The reversed signal $x(–t)$ is obtained as a reflection of $x(t)$ about the $t = 0$, i.e., we perform a reflection about the vertical axis. [Reflection about the horizontal axis results in $–x(t)$.] An example of this shown in Figure 1.35(a). In the figure, we get

$$x(t) = \begin{cases} -1, & -1 < t < 0 \\ 1 - t, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases} \tag{1.48}$$

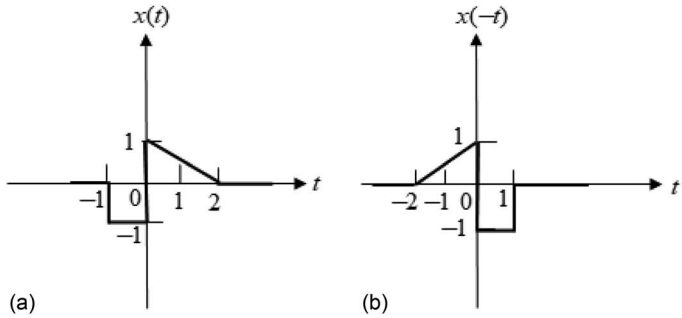


FIGURE 1.35 Time reversal of a signal.

From this, we obtain $x(-t)$ by replacing every t with $-t$ in eq. (1.48). Hence

$$x(-t) = \begin{cases} -1, & -1 < -t < 0 \\ 1-(-t), & 0 < -t < 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} -1, & 0 < t < 1 \\ 1+t, & -2 < t < 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.49}$$

as is evident from Figure 1.35(b).

1.6.2 Time Scaling

Time scaling involves the compression or expansion of a signal in time. Given a continuous-time signal $x(t)$, the time-scaled form of $x(t)$ is $x(at)$, where a is a constant. The scaled signal $x(at)$ will be compressed if $|a| > 1$ or expanded if $|a| < 1$; a negative value of a yields time reversal as well as compression or expansion. Again, we can obtain $x(at)$ from the signal in eq. (1.41) by replacing every t with at and simplifying. We show an example in Figure 1.36. Notice that the time reversal discussed in Subsection 1.6.1 can be considered a special case of time scaling with the scaling factor $a = -1$.

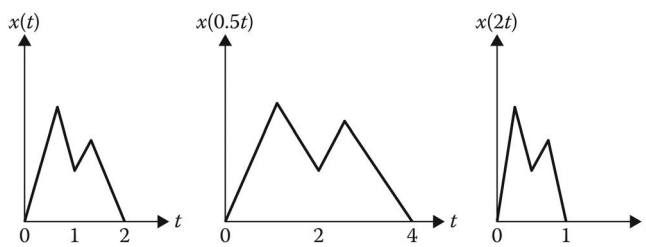


FIGURE 1.36 An example of time scaling.

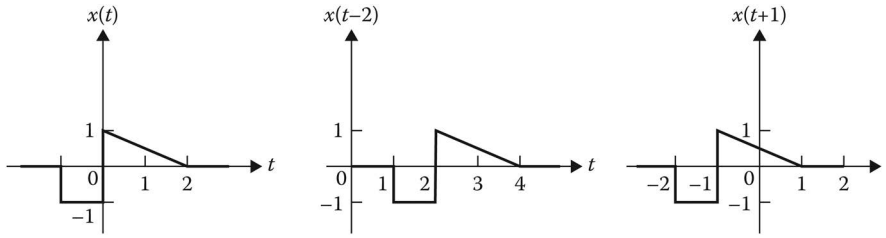


FIGURE 1.37 Time-shifted signals.

1.6.3 Time Shifting

Given a continuous-time signal $x(t)$, the time-shift form of $x(t)$ is $x(t - t_o)$, where t_o is a constant. If $t_o > 0$, then the signal $x(t - t_o)$ is *delayed* and the signal is shifted to the right relative to $t = 0$. When $t_o < 0$, the signal $x(t - t_o)$ is an *advanced* replica of $x(t)$, with $x(t)$ shifted to the left. Examples of $x(t - 2)$ and $x(t + 1)$ are shown in Figure 1.37.

For any signal $x(t)$, the transformation $at + b$ on the independent variable can be performed as follows:

$$x(at+b) = x\big(a(t+b/a)\big) \tag{1.50}$$

where a and b are constants. This involves two steps:

1. Scale by factor a. If a is negative, reflect x(t) about the vertical axis.
2. If b/a is negative, shift $x(t)$ to the right. If b/a is positive, shift $x(t)$ to the left.

1.6.4 Amplitude Transformations

The previous three transformations deal with the independent variable t. Equivalent transformations of the amplitude of the signal will now be discussed. Given a signal $x(t)$, amplitude transformations take the general form

$$y(t) = Ax(t) + B \tag{1.51}$$

where A and B are constants. For example, let

$$y(t) = -2x(t) + 4 \tag{1.52}$$

We notice that $A = -2$ means amplitude reversal [$-x(t)$ implies reflection about the horizontal axis] and amplitude scaling ($|A| = 2$). Also, $B = 4$ shifts vertically the amplitude of the signal.

Example 1.10

A continuous-time signal is shown in Figure 1.38. Sketch each of the following signals:

a. $x(2t - 6)$, (b) $x(t/2 + 1)$, (c) $y(t) = -1 + 2x(t)$

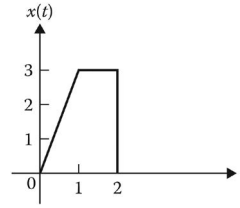


FIGURE 1.38 For Example 1.10.

SOLUTION

a. We can do this in two ways:

Method 1: We can write this as $x(2(t-3))$, indicating that $x(t)$ is compressed by a factor of 2 and shifted to the right by 3 units. Hence $x(2t-6)$ is as sketched in Figure 1.39(a).

Method 2: From Figure 1.38,

$$x(t) = \begin{cases} 3t, & 0 < t < 1 \\ 3, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases} \tag{1.10.1}$$

Replacing every t with $2t-6$ gives

$$x(2t-6) = \begin{cases} 3(2t-6), & 0 < 2t-6 < 1 \\ 3, & 1 < 2t-6 < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 6t-18, & 3 < t < 3.5 \\ 3, & 3.5 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

which is what we have in Figure 1.39(a).

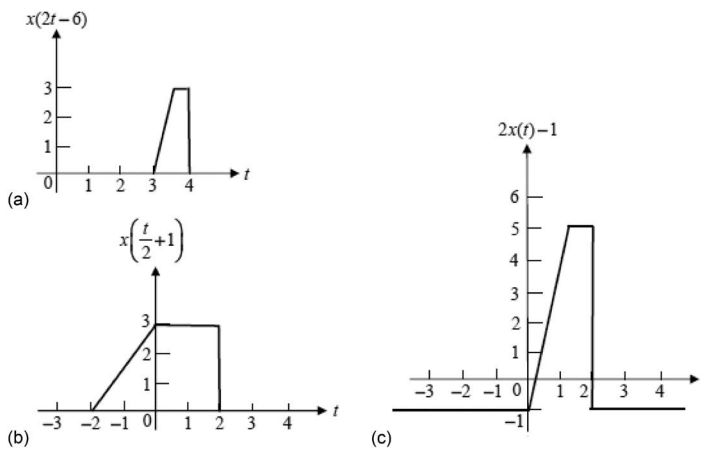


FIGURE 1.39 For Example 1.10.

b. *Method 1*: We write $x(t/2 + 1) = x(1/2(t + 2))$ indicating that $x(t)$ is expanded by a factor of 2 and advanced or shifted left by 2 units. The signal $x(t/2 + 1)$ is sketched in Figure 1.39(b)

Method 2: Replace every t in eq. (1.8.1) by $t/2 + 1$.

$$x(t/2+1) = \begin{cases} 3(t/2+1), & 0 < t/2+1 < 1 \\ 3, & 1 < t/2+1 < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1.5t+3, & -2 < t < 0 \\ 3, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

which agrees with the sketch in Figure 1.39(b).

c. This deals with amplitude amplification; there is no transformation of time t.

Method 1: The signal $x(t)$ is amplified by 2 and lowered by –1, as shown in Figure 1.39(c), giving $-1 + 2x(t)$.

Method 2:

$$y(t) = -1 + 2x(t) = \begin{cases} -1+6t, & 0 < t < 1 \\ -1+6, & 1 < t < 2 \\ -1, & \text{otherwise} \end{cases}$$

$$= \begin{cases} -1+6t, & 0 < t < 1 \\ 5, & 1 < t < 2 \\ -1, & \text{otherwise} \end{cases}$$

which agrees with what we have in Figure 1.39(c).

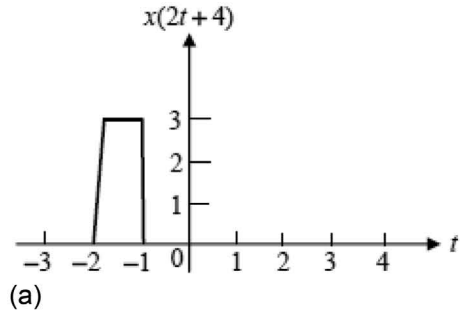
Practice Problem 1.10 With the continuous-time signal in Figure 1.38, sketch each of the following signals:

a. $x(2t + 4)$, (b) $-2x(t - 1)$, (c) $2 + x(-t)$.

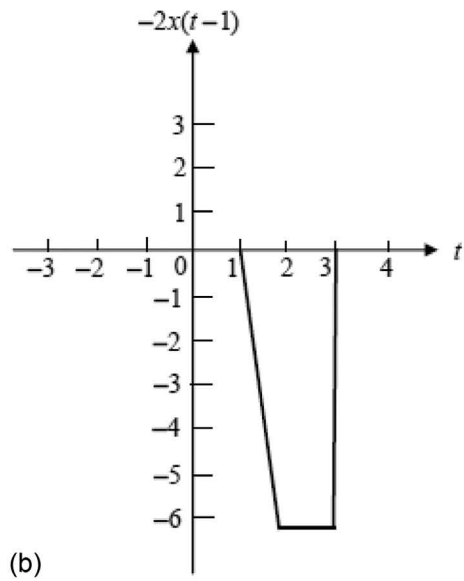
Answers: See Figure 1.40.

1.7 BASIC OPERATIONS ON DISCRETE SIGNALS

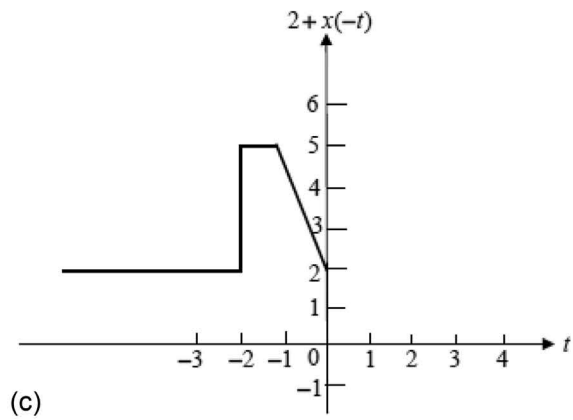
Operations of discrete times are basically extensions of the continuous-time signal operations. Operations on signals affect the time parameter of the signal. These operations are used in many signal-processing applications. We concentrate on the basic signal operations, which manipulate the signal characteristics by acting on



(a)



(b)



(c)

FIGURE 1.40 For Practice Problem 1.10.

the independent variables. This means that instead of performing operations like addition, subtraction, and multiplication between signals, we will perform them on the independent variable. We will consider the following signal operations: time shifting, time reversal, and time scaling.

1.7.1 Time Shifting

Time shifting refers to the shifting of a signal in time. This is done by adding or subtracting an integer quantity of the shift to the time variable in the function. Subtracting an integer quantity from the time variable will shift the signal to the right (delay). Adding an integer quantity to the time variable will shift the signal to the left (advance). An example is shown in Figure 1.41. As another example, let

$$x[n] = \begin{cases} 2, & n = -2 \\ 4, & n = -1 \\ 3, & n = 0 \\ -4, & n = 2 \\ 1, & n = 3 \\ 0, & n = \text{otherwise} \end{cases}$$

This can be expressed in terms of impulse sequences as

$$x[n] = 2\delta[n+2] + 4\delta[n+1] + 3\delta[n] - 4\delta[n-2] + \delta[n-3]$$

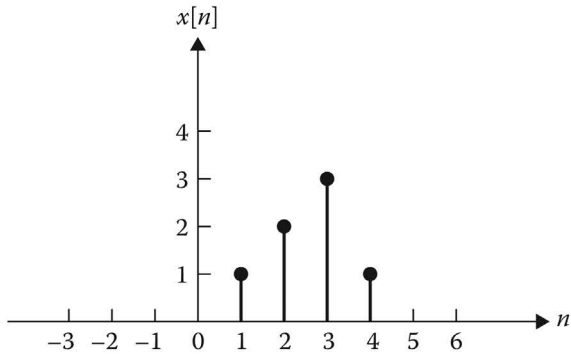
This can also be illustrated graphically as shown in Figure 1.42.

1.7.2 Time Reversal

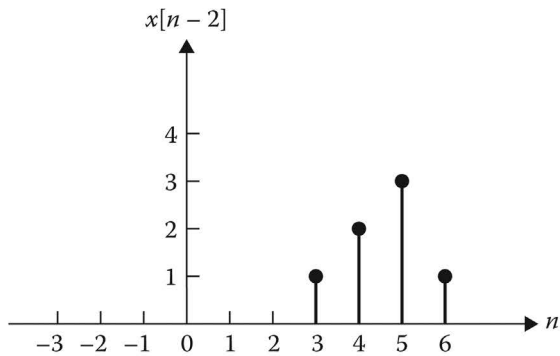
This operation causes the original signal to flip along its vertical axis of reference. As a result, the operation is known as the time reversal or time reflection of the signal. It requires replacing $x[n]$ with $x[-n]$. Time-reversal operations are used when computing the convolution of signals. An example is shown in Figure 1.43.

1.7.3 Expansion

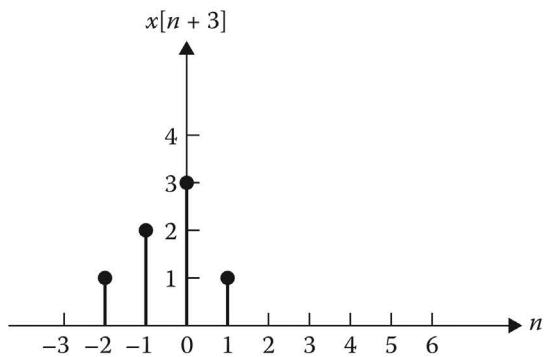
We can expand or compress a signal $x[n]$ to get $x[nk]$ for compression or $x[n/m]$ for expansion. In compression, we multiply the index n by k to get $x[nk]$, which extracts every kth sample of $x[n]$. The signal is shorter, and intermediate samples are lost. In expansion, the signal $x[k]$ is expanded by dividing the index n by an integer m to produce $x[n/m]$. This produces every mth sample. The signal becomes longer, and intermediate samples should be set to zero or interpolated. In the expanded signal we can only place the entries of the original signal $x[n]$ at values of n that are multiples of m. We fill in the undetermined values with zeros. An example is shown in Figure 1.44.



(a)



(b)



(c)

FIGURE 1.41 Shifting of signal $x[n]$. (a) Original signal $x[n]$. (b) Delayed signal $x[n - 2]$. (c) Advanced signal $x[n + 3]$.

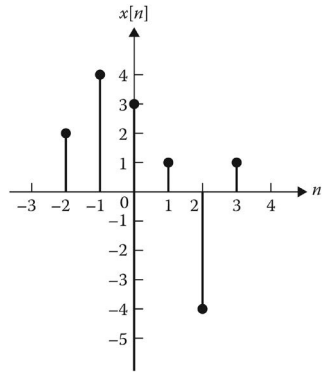


FIGURE 1.42 Graph of signal $x[n]$.

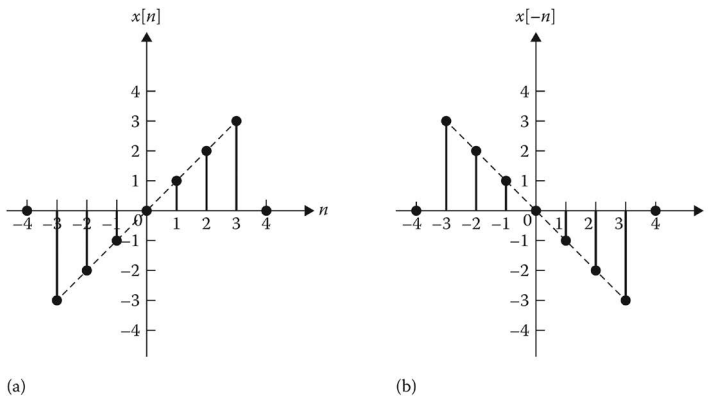


FIGURE 1.43 Time reversal of $x[n]$.

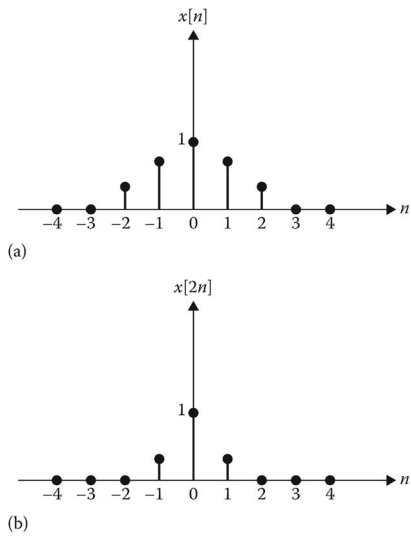


FIGURE 1.44 The signal $x[n]$ is expanded by 2.

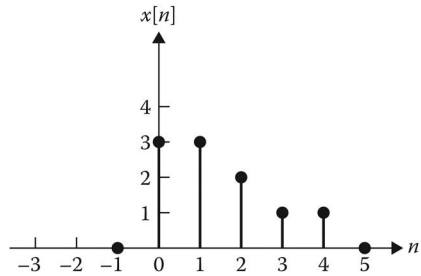


FIGURE 1.45 For Example 1.11.

Example 1.11

A discrete-time signal is shown in Figure 1.45. Sketch each of the following signals: (a) $x[n-3]$, (b) $x[-n+3]$, (c) $x[2n]$.

SOLUTION

a. This involves shifting $x[n]$ 3 units to the right, as shown in Figure 1.46(a).
b. We can write $x[-n+3] = x[-(n-3)]$, meaning that we perform a reflection about the vertical axis to get $x[-n]$ and we then shift this 3 units to the right. This is shown in Figure 1.46(b).
c. This involves compressing $x[n]$ by a factor of 2, as shown in Figure 1.46(c).

Practice Problem 1.11 With the discrete-time signal shown in Figure 1.45, sketch each of the following signals: (a) $x[-n]$, (b) $x[n+2]$, (c) $x[n/2]$.

Answers: See Figure 1.47.

Example 1.12

a. Let $x[n] = \delta[n] - 2\delta[n-1] + \delta[n-3]$.
Sketch $x[n]$.
b. Determine and sketch

$$y[n] = x[n+1] + x[n] + x[n-2]$$

SOLUTION

a. $x[n]$ is shown in Figure 1.48(a).
b. $y[n] = x[n+1] + x[n] + x[n-1]$
$= \delta[n+1] - 2\delta[n] + \delta[n-2] +$
$+ \delta[n] - 2\delta[n-1] + \delta[n-3] +$
$+ \delta[n-2] - 2\delta[n-3] + \delta[n-5]$
$= \delta[n+1] - \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3] + \delta[n-5]$.

$y[n]$ is sketched in Figure 1.48(b).

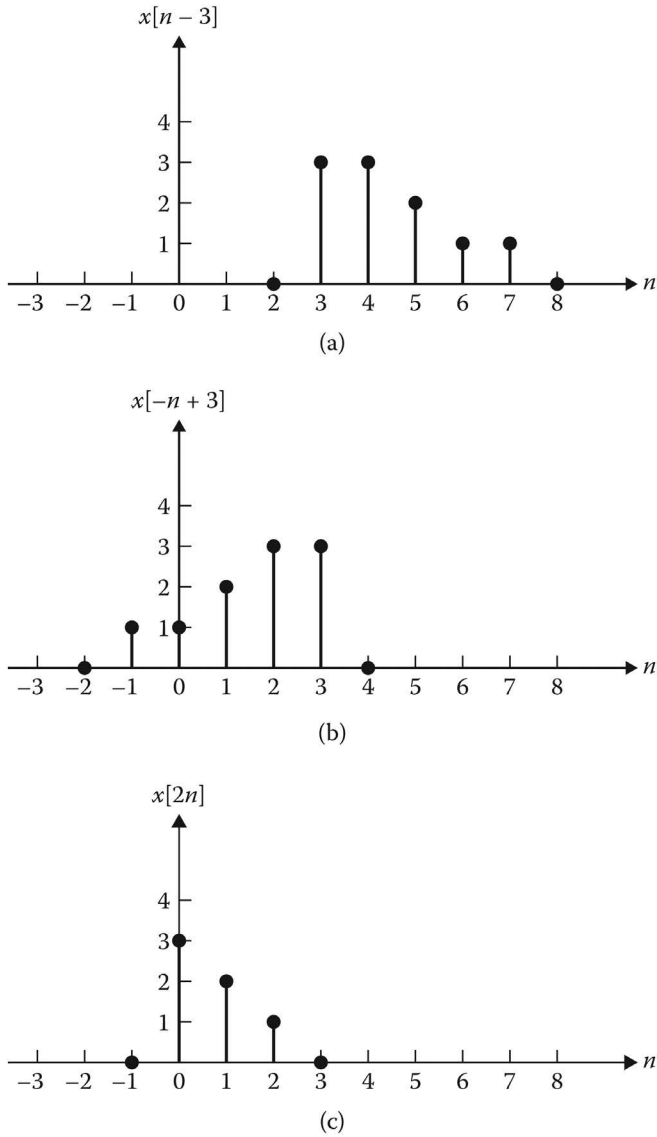


FIGURE 1.46 For Example 1.11.

Practice Problem 1.11 Given the discrete-time signal $y[n]$ in Figure 1.49, (a) express $y[n]$ in terms of the impulse sequences.

a. Sketch $x[n] = y[n - 3]$
b. Sketch $z[n] = 2y[n + 1]$

Answers: (a) $y[n] = \delta[n] + 2\delta[n - 1] - \delta[n - 2] + \delta[n - 3]$; for (b) and (c), see Figure 1.50.

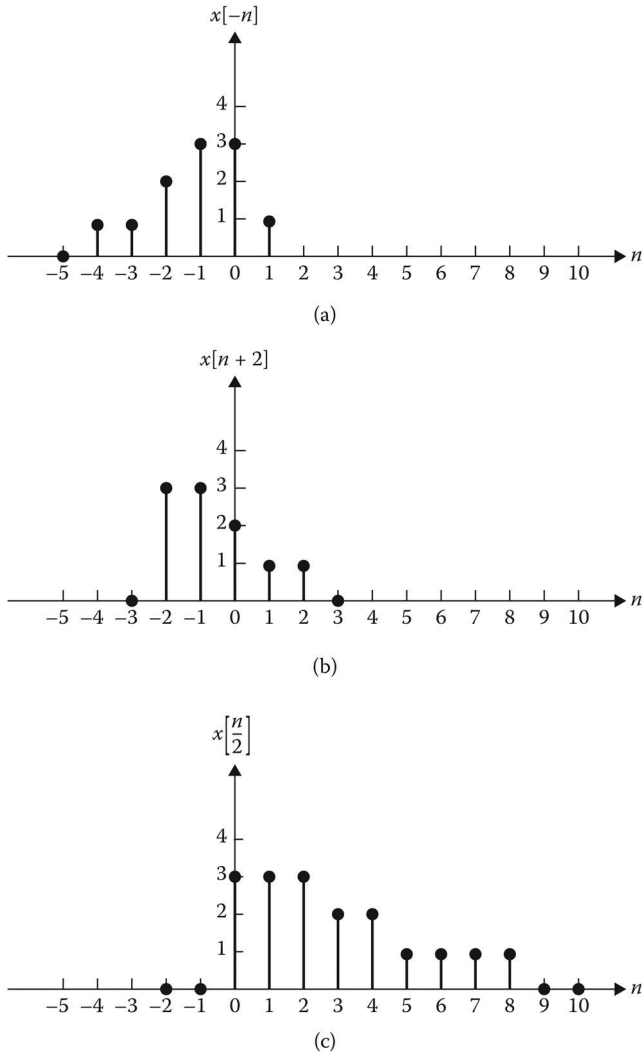


FIGURE 1.47 For Practice Problem 1.11.

1.8 CLASSIFICATION OF SYSTEMS

As mentioned in Section 1.2, a system may be regarded as a mathematical model of a physical process that relates the input signal $x(t)$ to the output signal $y(t)$. There are several ways to represent a system. Two common ways are: (1) using a block diagram and (2) using a differential equation for a continuous system or a difference equation for a discrete-time system. Using the fundamental building blocks (such as adders, multipliers, delay, etc.), the block diagram of a system can be prepared. For example, the relationship between input $x(t)$ and output $y(t)$ is illustrated in Figure 1.51(a). Although

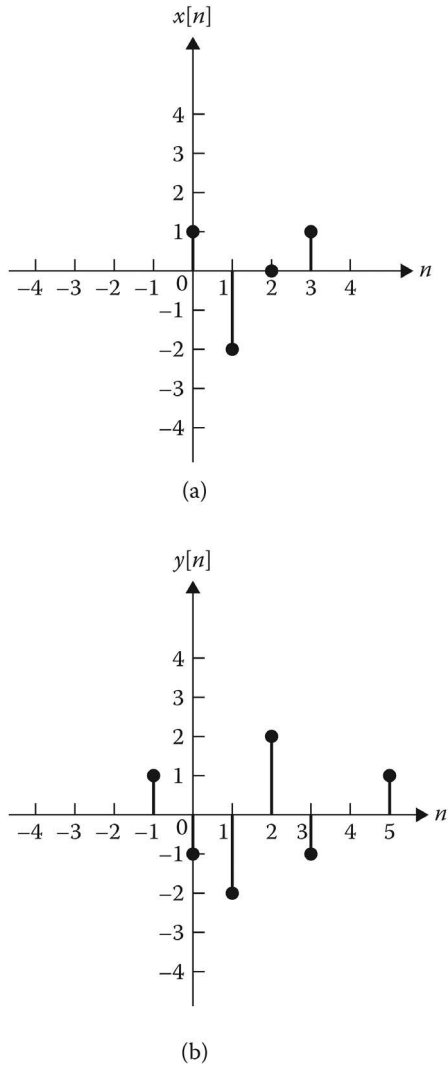


FIGURE 1.48 For Example 1.12.

a system may have many input and output signals, as shown in Figure 1.51(b), we will focus our attention on the single-input, single-output case in this book.

If x is the input signal and y is the output signal, they are related through a transformation

$$y = \mathbf{T}x \tag{1.53}$$

where $\mathbf{T}$ is an operator transforming x into y.

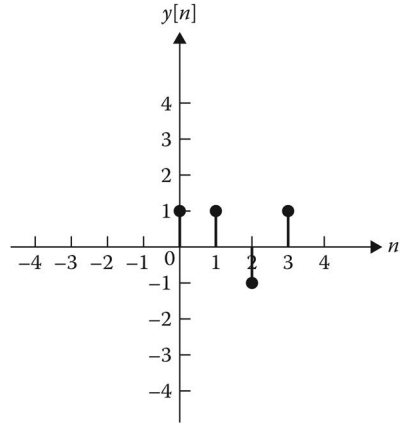


FIGURE 1.49 For Practice Problem 1.12.

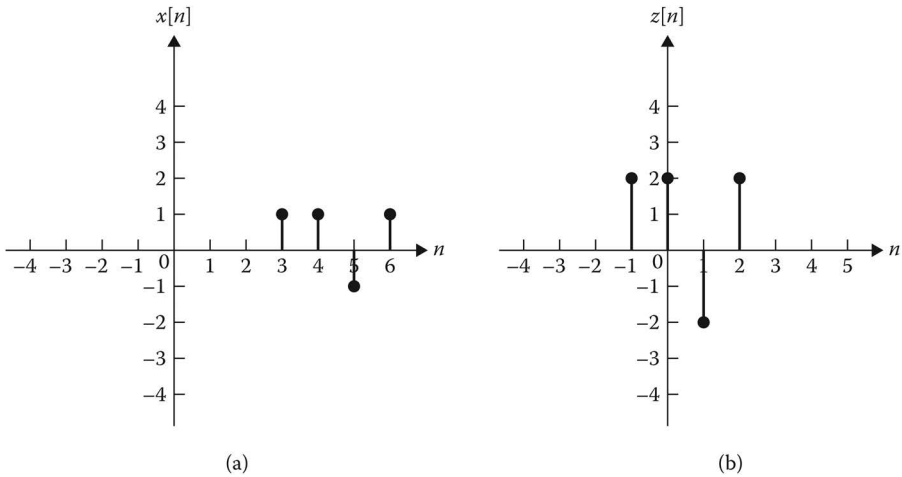


FIGURE 1.50 For Practice Problem 1.12.

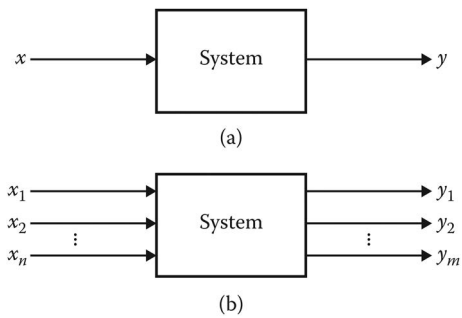


FIGURE 1.51 Block diagram representation: (a) system with single input and single output and (b) system with many inputs and outputs.

For example, a continuous-time system is represented by the ordinary differential equation

$$\frac{d^2y(t)}{dt^2}+\frac{dy}{dt}+2y(t)=x(t),\quad t>0$$

where $x(t)$ is the input and $y(t)$ is the output. In the same way, the following difference equation

$$y(n)=y(n-1)+x(n)-2x(n-1)+x(n-2)$$

represents a discrete-time system.

1.8.1 Continuous-Time and Discrete-Time Systems

A system is continuous-time if the input and output signals are continuous-time. It is discrete-time if the input and output signals are discrete-time. In continuous-time systems, time is measured continuously. Some continuous-time systems are described by ordinary differential equations, algebraic equations (resistive circuits), polynomial equations (diodes), integral equations (op amp integrator circuits), etc. For discrete-time systems, time is defined only at discrete instants, and the systems are described by difference equation and any other way the input-output properties of the system may be specified. Continuous-time and discrete-time systems are illustrated in Figure 1.52.

1.8.2 Causal and Noncausal Systems

A signal (t) is a causal signal if the signal $(t) = 0$ for $t < 0$. That is, a causal signal does not exist for negative time. A good example of a causal signal is the unit step signal $u(t)$. The output of a system may depend on the present and past inputs. A system that has this property is known as a *causal system*. A causal (or nonanticipatory) system

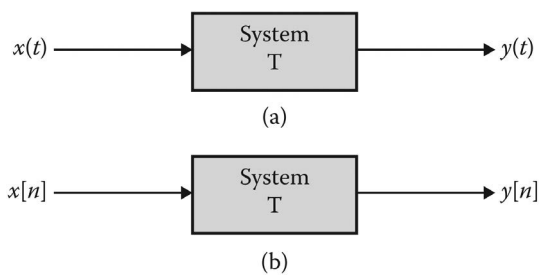


FIGURE 1.52 Block diagram representing (a) a continuous-time system and (b) a discrete-time system.

is one whose output $y(t)$ at the present time depends only on the present and past values of the input $x(t)$.

A **causal system** is one whose present response depends only on the present and the past inputs, but not on the future values of the input.

Examples of causal systems are:

$$y(t) = x(t-1) \tag{1.54}$$

$$y[n] = x[n/2] \tag{1.55}$$

Every physical system is causal. The motion of a car is causal since it does not expect the future actions of the driver. Causality is a necessary condition for a system to be realized in the real world we live in.

If a system is not causal, it is said to be *noncausal* or anticipatory. Typical examples of noncausal systems are

$$y(t) = x(t+2) \tag{1.56}$$

$$y[n] = x[1-n] \tag{1.57}$$

The system in eq. (1.56) is not causal because the output at, say, $t = 0$ is equal to input at $t = 2$. This is not physically realizable. The same logic applies to the system in eq. (1.57).

An ideal filter is noncausal and is not physically realizable; it cannot be built in practice. It should be noted that causality is not often an essential constraint in applications in which the independent variable is not time, such as in image processing.

1.8.3 Linear and Nonlinear Systems

Linearity is the property of the system describing a linear relationship between input (cause) and output (effect). The property is a combination of both the homogeneity (scaling) property and the additivity property.

The homogeneity property requires that if the input is multiplied by any constant k, then the output is multiplied by the same constant, i.e.,

$$\mathbf{T}\{kx\} = ky \tag{1.58}$$

The additivity property requires that the response to a sum of inputs is the sum of the responses to each input applied separately. If $\mathbf{T}x_1 = y_1$ and $\mathbf{T}x_2 = y_2$, then

$$\mathbf{T}\{x_1 + x_2\} = y_1 + y_2 \tag{1.59}$$

We can combine eqs. (1.58) and (1.59) as

$$\mathbf{T}\{k_1x_1 + k_2x_2\} = k_1y_1 + k_2y_2 \tag{1.60}$$

where k_1 and k_2 are constants.

A **linear system** meets the two conditions in eqs. (1.58) and (1.60); i.e., a system is linear if it satisfies homogeneity and additive properties.

A linear system may be regarded as one in which all the interrelationships among the quantities involved are expressed by linear equations, which may be algebraic, differential, or integral. A very important consequence of linearity is that *superposition principle* applies. If an input consists of the weighted sum of several signals, then the output is the weighted sum of the responses of the system to each of those signals.

An example of a linear system is a circuit that contains resistors, capacitors, and inductors, since these are linear elements. Devices such as rectifiers, diodes, and saturating magnetic devices are nonlinear. A system having even one such device is treated as nonlinear. Examples of nonlinear systems are:

$$y = \sin\ x \tag{1.61}$$

$$y = x^3 \tag{1.62}$$

1.8.4 Time-Varying and Time-Invariant Systems

When one or more parameters of the system vary with time, the system is said to be time-varying.

A **time-varying system** is one whose parameters vary with time.

Continuous-time, time-varying systems are described by time-varying differential equations. Discrete-time, time-varying systems are described by time-varying difference equations. If all system parameters are constant with time, the system is said to be time-invariant, or fixed.

In a **time-invariant system**, a time shift (advance or delay) in the input signal leads to the time shift in the output signal.

For a continuous-time system, the system is time-invariant when

$$T\{x(t-\tau)\} = y(t-\tau) \tag{1.63}$$

where τ is a constant. Consider the input $x(t)$ of a linear system yielding an output $y(t)$. For the system to be time-invariant, a shifted input $x(t - \tau)$ should result in a

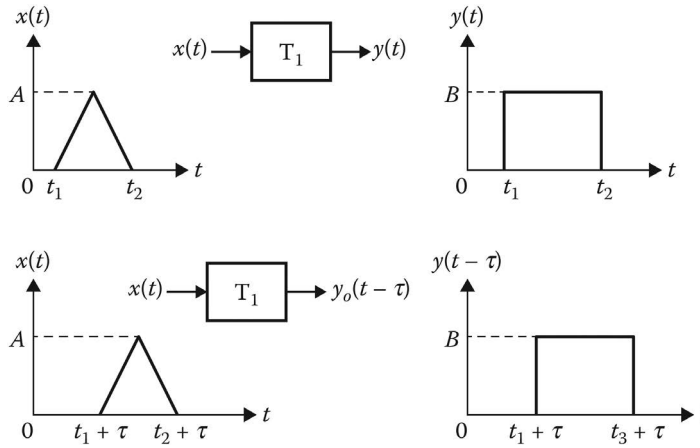


FIGURE 1.53 An example of a time-invariant system.

shifted output $y(t - \tau)$. Figure 1.53 shows an example of a time-invariant system, while Figure 1.54 gives an example of a time-varying system.

For a discrete-time system, the system is time-invariant when

$$T\{x[n-m]\} = y[n-m] \tag{1.64}$$

for any integer m. A continuous-time/discrete-time system that does not satisfy eq. (1.63) or (1.64) is time-varying. Such a system has parameters that vary with time. We will confine our efforts to linear time-invariant (LTI) systems in this book.

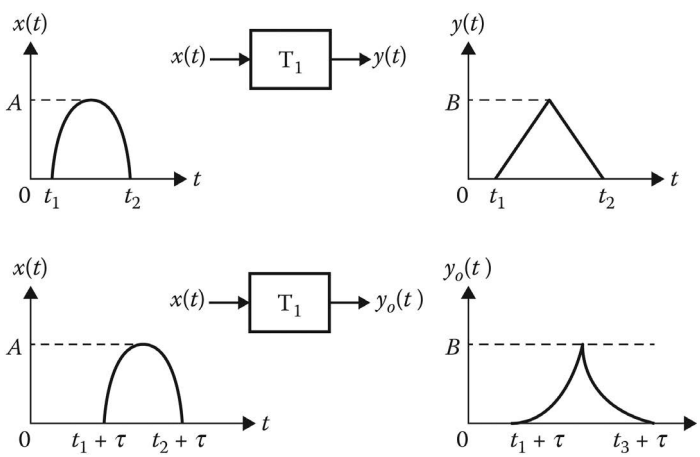


FIGURE 1.54 An example of a time-varying system.

1.8.5 Systems With and Without Memory

When the output of a system depends on the past and/or future input, the system is said to have memory. For example, a system described by

$$y(t) = x(t) - x(t-1) + x(t+2) \tag{1.65}$$

has memory since the output $y(t)$ requires the past input $x(t-1)$, the current input $x(t)$, and the future input $x(t+2)$.

If the output of a system does not depend on past and/or future input, the system is memoryless (or instantaneous).

A **memoryless system** is one in which the current output depends only on the current input; it does not depend on past or future inputs.

A system with memory is also called a *dynamic system*. A memoryless system is called a *static system*. Many practical systems such as restive networks, amplifiers, operational amplifiers, and diodes are usually modeled as memoryless systems.

Example 1.13

Show that the system represented by

$$\frac{dy}{dt} + 4y(t) = 2x(t) \tag{1.13.1}$$

is linear.

SOLUTION

To prove that the system is linear, we need to show that eq. (1.52) is satisfied. Let the inputs $x_1(t)$ and $x_2(t)$ cause the system responses $y_1(t)$ and $y_2(t)$, respectively; i.e.,

$$\frac{dy_1}{dt} + 4y_1(t) = 2x_1(t) \tag{1.13.2}$$

$$\frac{dy_2}{dt} + 4y_2(t) = 2x_2(t) \tag{1.13.3}$$

Multiplying eq. (1.13.2) by constant A and eq. (1.13.3) by constant B and adding, we get

$$\frac{d}{dt}\left[Ay_1 + By_2\right] + 4\left[Ay_1 + By_2\right] = 2\left[Ax_1 + Bx_2\right]$$

This satisfies the system equation (1.13.1) with

$$x = Ax_1 + Bx_2$$

$$y = Ay_1 + By_2$$

(The time dependence is dropped for simplicity.) In other words, with the input as $x = Ax_1 + Bx_2$, the response is $y = Ay_1 + By_2$. Superposition holds, and we conclude that the system is linear.

Practice Problem 1.13 Show that the system described by

$$y(t) = 2x(t) - 5$$

is nonlinear.

Answer: Proof; a part of the output signal does not depend on the input signal.

Example 1.14

A modulator with carrier frequency of ω_c gives an output

$$y(t) = x(t)\cos\omega_c t$$

Determine whether the system is: (a) memoryless, (b) causal, (c) time-invariant.

SOLUTION

a. The system is memoryless since the output $y(t)$ depends only on the present values of the input $x(t)$; i.e., the output is a function of the input at only the present time.
b. It is causal since the output $y(t)$ does not depend on the future values of the input $x(t)$. Since it is memoryless, it is also causal.
c. To be time-invariant, an input $x(t - T)$ should produce an output $y(t - T)$. Let the input signal $x_o(t - t_o)$ produce an output signal y_o. Then

$$y_o(t) = x(t - t_o)\cos\omega_c t$$

But we know that

$$y(t - t_o) = x(t - t_o)\cos\omega_c(t - t_o) \neq y_o$$

The system fails to satisfy the time-invariance property of eq. (1.55). Hence, it is not time-invariant. It is time-varying.

Practice Problem 1.14 Consider the system represented by $y(t) = x(t - 3)$. Discuss the properties of the system; i.e., check whether it is memoryless, causal, and time-invariant.

Answers: Not memoryless, causal, and time-invariant.

1.9 APPLICATIONS

The information covered in this chapter is useful in many systems such as mass-spring-damper systems, moving average filters, stock trading, electrical circuits, square-law devices, and digital signal processing (DSP). We will briefly consider three simple applications related to what we have covered in this chapter. These applications deal with electric circuits, square-law devices, and DSP systems.

1.9.1 Electric Circuits

Electric circuits are mostly linear systems. Consider the one shown Figure 1.55. Our objective is to find the system equation for it. By applying Kirchhoff's voltage law, we obtain

$$x(t) = Ri(t) + y(t) \tag{1.66}$$

But $i(t) = \frac{1}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda$. Substituting this in eq. (1.66),

$$y(t) = x(t) - \frac{R}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda \tag{1.67}$$

Or we may differentiate this to get

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{dx}{dt} \tag{1.68}$$

This is an explicit input-output relationship between $x(t)$ (source voltage) and $y(t)$ (inductor voltage).

1.9.2 Square-Law Devices

A square law device is a device where either current or voltage depends on the square of the other. The input/output relationship is given by

$$y(t) = x^2(t) \tag{1.69}$$

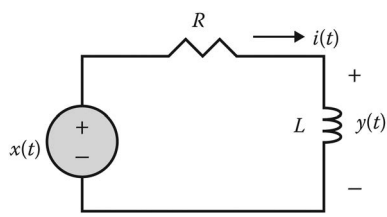


FIGURE 1.55 An electric circuit.

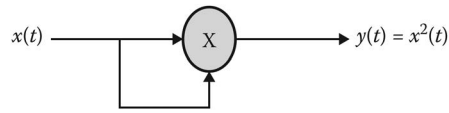


FIGURE 1.56 A square-law detector.

Any system that is defined by eq. (1.69) is called a *square-law device*. To realize this system, we use a signal amplifier, as shown in Figure 1.56. For this reason, the square-law detector is also called a *signal multiplier*. It can be built using operational amplifiers and diodes. It is evident from eq. (1.69) that the system is nonlinear and memoryless. A square-law device is often found in the receiver front end of radar and communication devices.

1.9.3 DSP Systems

Since most signals are analog in nature, an analog signal can be processed directly in its analog form, as shown in Figure 1.57. DSP provides an alternative means of processing the analog signal, as shown in Figure 1.58. DSP techniques are replacing analog signal processing in many fields such as telecommunications; radar and sonar; digital control; speech, audio, and video processing; and biomedicine. One main reason for preferring DSP is that it allows programmability, which means that the same DSP hardware can be used for different applications.

A DSP is a special microprocessor designed for digital signal processing. Most DSPs can be used to manipulate different kinds of information including sound, images, and video. The goal of a DSP is to process real-world analog signals. As illustrated in Figure 1.58, the first step is to convert the analog signal to a digital signal using an ADC. The ADC quantizes the sampled analog signal and codes the quantized signal into an acceptable format. It is common for the sequence coming from the ADC to be converted back to analog to drive a motor, a stereo system, or whatever. The conversion of the digital sequence to an analog signal requires a digital-to-analog converter (DAC). Most PCs include ADCs and DACs in their sound card. In these modern days, DSP systems such as high-speed modems and cell phones are common.

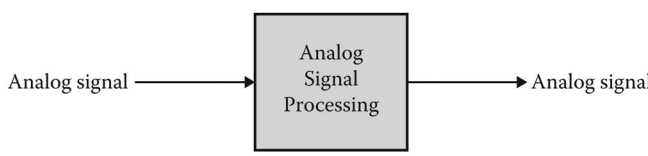


FIGURE 1.57 Analog signal processing.

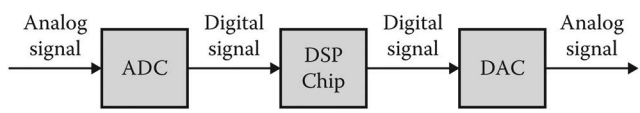


FIGURE 1.58 Digital signal processing.

1.10 COMPUTING WITH MATLAB®

A brief introduction to MATLAB® is given in Appendix B. The reader is encouraged to read Appendix B before proceeding with this section.

MATLAB provides built-in functions for unit step function $u(t)$ and unit impulse function $\delta(t)$. The unit step function is called **Heaviside** or **stepfun**, while the impulse function is **dirac**. Heaviside(t) is zero when $t < 0$, 1 for $t > 0$, and 0.5 for $t = 0$. **stepfun**(t,$t0$) returns a vector of the same length at t with zeros for $t < t0$ and ones for $t > t0$. **stepfun**(n,$n0$) works the same way for discrete signals. **dirac**(t) is zero for all t, except $t = 0$, where it is infinite.

MATLAB treats continuous-time signals differently from discrete-time signals. We will treat them separately in the following two examples.

Example 1.15

Use MATLAB to plot:

a. The signal $x(t) = 10e^{-0.5t}\sin(4\pi t) + 20e^{-t}\sin(2\pi t)$
b. The step function starting at $t = 1$ and going to the right

SOLUTION

a. The MATLAB code is shown here.

```
t= 0.1:2*pi/100:2*pi  % t ranges from 0.1 to 2*pi
x =10* exp(-0.5*t).*sin(4*pi*t) + 20*exp(-t).*sin(2*pi*t);
% generates values for x
plot(t,x);  %plot the signal
xlabel( 't in seconds')
ylabel('signal x(t)')
```

We let t vary from 0.1 to 2π seconds, with an increment of $2\pi/100$ second. Note the use of the. * operator. The terms 10*exp(–0.5*t), 20*exp(–t).* are all vectors. We use. * instead of * because we want the components of these vectors to be multiplied by their corresponding components of the other vectors. Thus, element-by-element operations require a dot before the operator. The plot is presented in Figure 1.59(a).

b. The MATLAB script for this case is presented next.

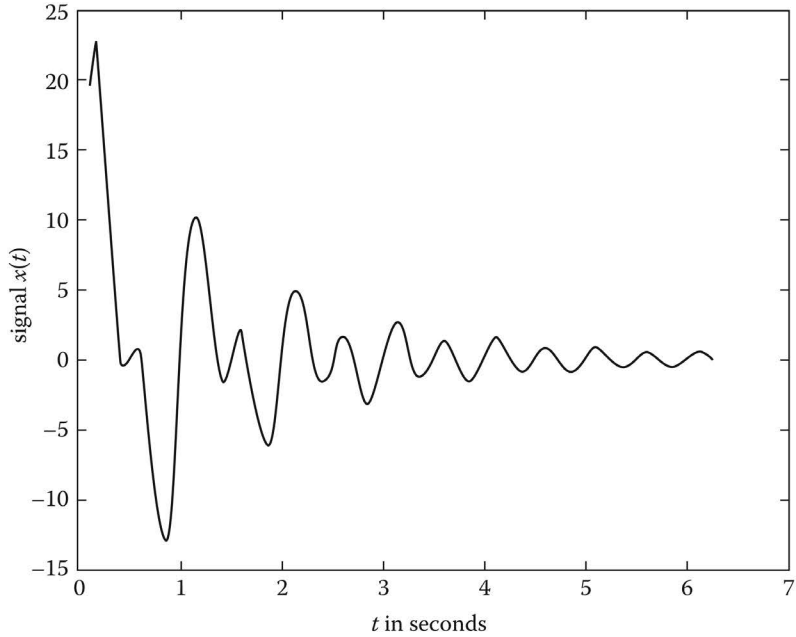
```
t = 0:0.1:5
t0 = 1;
u = stepfun(t,t0);
plot(t,u)
```

As shown in Figure 1.59(b), the plot is a delayed unit step, i.e., $u(t - 1)$.

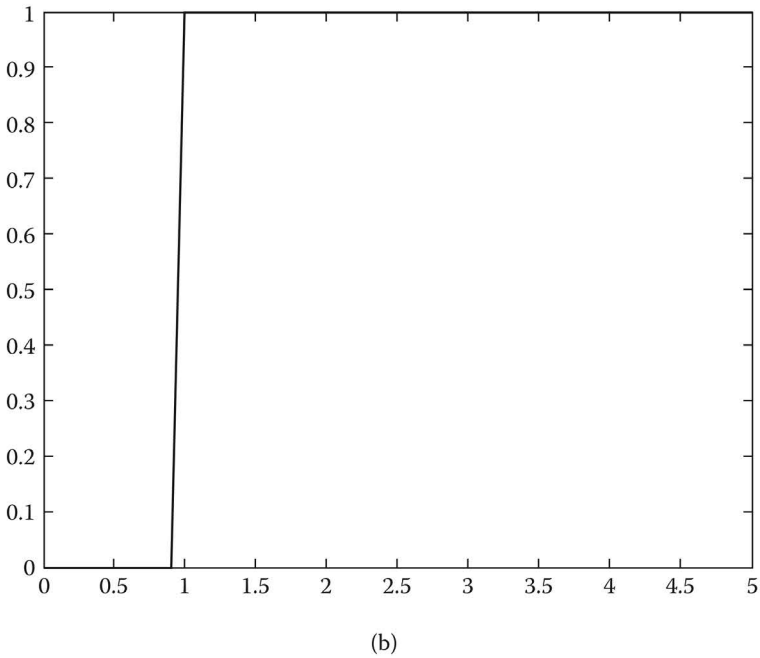
Practice Problem 1.15 Using MATLAB, plot the signal

$x(t) = e^{-0.2t}\cos(0.5t + 1.2)$, $0 < t < 20$ seconds.

Answer: See Figure 1.60.



(a)



(b)

FIGURE 1.59 (a) For Example 1.15(a). (b) For Example 1.15(b).

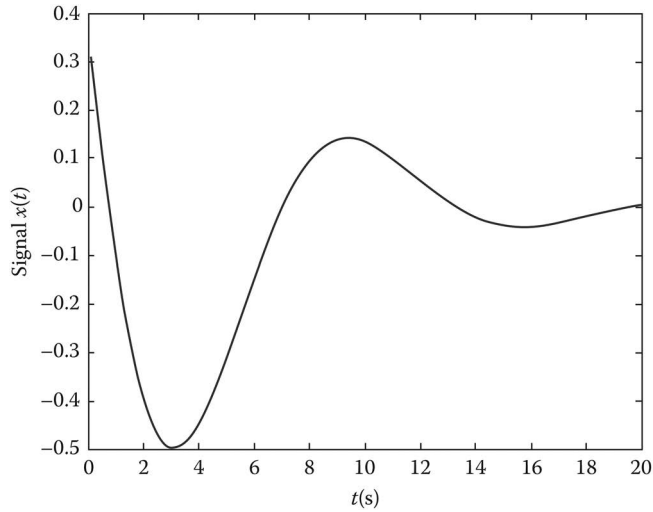


FIGURE 1.60 For Practice Problem 1.15.

Example 1.16

Using MATLAB, plot the following discrete signals:

a. $x[n] = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ -2, & n = 3 \\ 2, & n = 4 \\ 0, & \text{otherwise} \end{cases}$

b. $f[n] = A(\alpha)^n$ with $A = 5, \alpha = -0.6$

SOLUTION

a. A plot of this discrete signal is generated by the following MATLAB commands:

```
n=-3:1:5;    %  n = -3, -4,…,4,5
x=[ 0 0 0 1 3 0 -2 2 0 ];
stem(n,x);
xlabel('n');
ylabel( 'x[n]');
  grid;
```

The plot is shown in Figure 1.61(a).

b. The MATLAB code for generating and plotting the discrete signal is as follows:

```
n=-5:1:5;
f=5*(-0.6).^n;
```

```
stem(n,f);
xlabel('n');
ylabel('f(n)');
```

The plot is shown in Figure 1.61(b). Notice that $f[n]$ is an alternating exponential sequence.

Practice Problem 1.16 Use MATLAB to plot the following discrete signals:

a. $y[n] = 10\sin\left(\frac{2\pi n}{16}\right)$ for $n = 0$ to 32

b. The exponential sequence in Example 1.12(b) multiplied by a step sequence

Answer: See Figure 1.62.

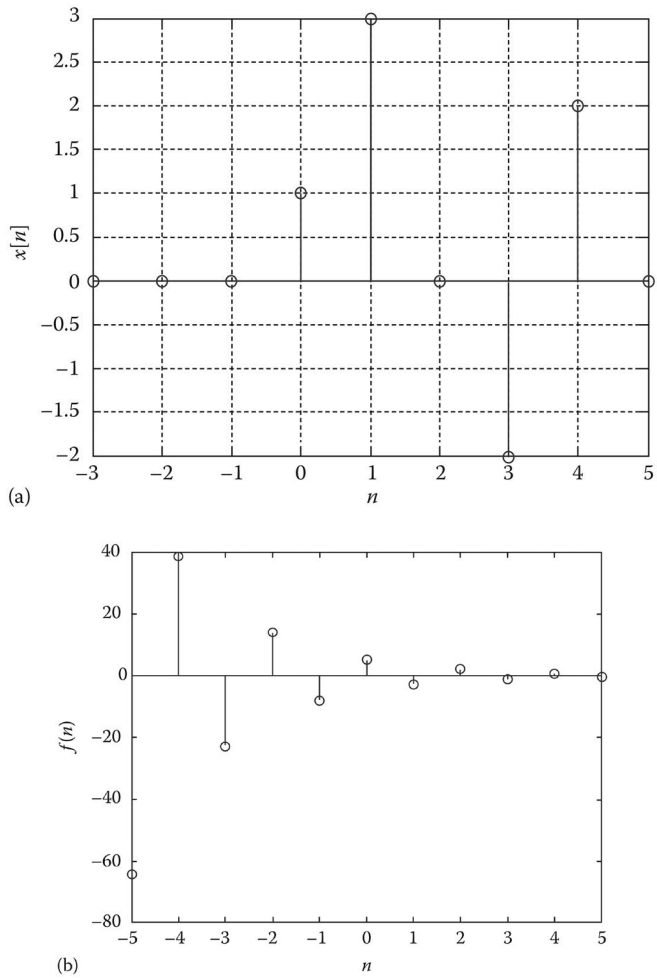


FIGURE 1.61 (a) For Example 1.15(a). (b) For Example 1.15(b).

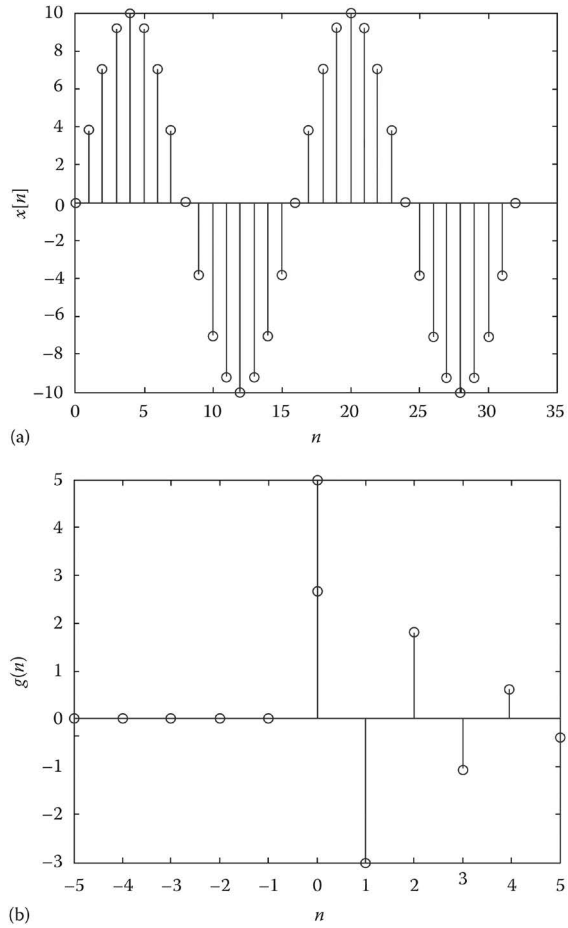


FIGURE 1.62 (a) For Practice Problem 1.16(a). (b) For Practice Problem 1.16(b).

SUMMARY

1. A signal is a set of data or functions that represents a variable of interest.
2. A system is a device, process, or algorithm that operates on an input signal to yield an ignal.
3. A signal may be classified in many ways:
   (a) A *continuous-time* signal $x(t)$ has a value specified at all times, whereas a *discrete-time* signal $x[n]$ is specified only at a finite set time instant.
   (b) A *periodic* signal repeats itself after period T, i.e., $x(t + T) = x(t)$. An *aperiodic* signal is one that is not periodic.
   (c) An *analog* signal is one that can take on any value over a continuum; a *digital* signal can only take a finite number of values.
   (d) A signal $x(t)$ or $x[n]$ is an *energy* signal if and only if $0 < E < \infty$ and consequently $P = 0$; i.e., it has finite energy. A signal $x(t)$ or $x[n]$ is a *power* signal if and only if $0 < P < \infty$ and consequently $E = \infty$; it has finite power.
   (e) A signal is *even* if $x(t) = x(–t)$; it is *odd* if $x(t) = –x(–t)$.

4. Examples of standard signals are the singularity functions, which include the *unit impulse* $\delta(t)$ or $\delta[n]$, the *unit step* $u(t)$ or $u[n]$, and the *unit ramp* $r(t)$ or $r[n]$. The shifting property of the impulse function is

$$\int_a^b f(t)\delta(t-t_o)\,dt = f(t_o)$$

   In addition to the singularity functions, we considered the unit rectangular function $\Pi(t/\tau)$, the unit triangular function $\Lambda(t/\tau)$, the sinusoidal function, and the exponential function.
5. The following operations can be performed on a signal:
   (a) A signal $x(-t)$ is the time reversal of $x(t)$; it is obtained by reflecting $x(t)$ about $t = 0$.
   (b) The signal $x(at)$ is compressed if $a > 1$ or expanded if $a < 1$.
   (c) The signal $x(t - t_o)$ is delayed by t_o seconds, while the signal $x(t + t_o)$ is advanced by t_o seconds, assuming $t_o > 0$.
   (d) Amplitude transformations take the general form of $y(t) = Ax(t) + B$.
6. A system may be classified in many ways:
   (a) A system is *continuous-time* if the input and output signals are continuous-time; it is *discrete-time* if the input and output signals are discrete-time.
   (b) A system is *causal* if its present response does not depend on the future values of the input; it is *noncausal* otherwise.
   (c) A system is *linear* if it is characterized by the linearity property; it is *nonlinear* if superposition does not hold.
   (d) A system is *time-varying* if its parameters change with time; it is *time-invariant* if its parameters do not change with time.
   (e) A *memoryless system* is one in which the current output depends only on the current input; a system with memory is one whose output depends on the present and past or future inputs.
7. Specific applications considered in this chapter are electric circuits, square-law devices, and DSP systems.
8. MATLAB can be used to process continuous-time and discrete-time signals.

REVIEW QUESTIONS

1.1 Which of the following is not a signal?
(a) voltage, (b) blood, (c) torque, (d) pressure, (e) displacement

1.2 Any set of mathematical relationships between input and output variables constitutes a system.
(a) True, (b) False

1.3 Which of the following is not a system?
(a) automobile speed, (b) a circuit, (c) a camera, (d) a computer program

1.4 The exponential signal is:
(a) a power signal, (b) an energy signal, (c) neither a power or an energy signal

1.5 The signals $\delta(t)$, u(t), and r(t) are known as:
(a) transformation functions, (b) expansion functions, (c) singularity functions, (d) basic functions

1.6 Which of these is not true?
(a) $t^2\delta(t) = 0$
(b) $\cos(t)\delta(t) = \delta(t)$
(c) $\sin(t)\delta(t-\pi) = \delta(t-\pi)$
(d) $\int_{-\infty}^{\infty} (2t^2+1)\delta(t)\,dt = 1$

1.7 Given the signal $x(t)$, which of the following is true of the signal $x(2t + 8)$?
(a) $x(t)$ is shifted to the left by 8 units
(b) $x(t)$ is compressed by a factor of 2 and then shifted left by 4 units
(c) $x(t)$ is expanded by a factor of 2 and then shifted right by 4 units
(d) $x(t)$ is reflected about the vertical axis and then shifted

1.8 A system is described by $y(t) = \cos x(t) + 2\sin x(t - 1)$. It is:
(a) causal, (b) noncausal, (c) memoryless, (d) with memory

1.9 Causal systems are also referred to as physically realizable systems.
(a) True, (b) False

1.10 The impulse function in MATLAB is:
(a) Heaviside, (b) stepfun, (c) dirac, (d) stem

Answers: 1.1(b), 1.2(a), 1.3(a), 1.4(c), 1.5(c), 1.6(c), 1.7(b), 1.8(a, d), 1.9(a), 1.10(c).

PROBLEMS

Section 1.3: Classification of Signals

1.1 Show that the complex sinusoidal signal $x(t) = Ae^{j\omega_o t}$ is periodic.

1.2 Find the energy content of

$$x(t) = \begin{cases} e^{-3t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and determine whether it is a power or energy signal.

1.3 Let

$$x(t) = \begin{cases} e^{-2t}\sin t, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Calculate the energy content of the signal.

1.4 Find out if the following signals are power or energy signals or neither.
(a) $x(t) = 10\sin 2\pi t$
(b) $y(t) = 2[u(t) - u(t - 4)]$
(c) $z(t) = r(t) - r(t - 2)$
(d) $h(t) = 10e^{2t}$

1.5 Repeat Problem 1.4 for the following signals.
(a) $x(t) = e^{-3t}\,u(t)$
(b) $x(t) = (e^{-2t} + 1)u(t)$
(c) $x(t) = t^{-1/2}\,u(t - 2)$

Section 1.4: Basic Continuous-Time Signals

1.6 The signal $x(t)$ in Figure 1.63 is called the gate function.
(a) Express $x(t)$ in terms of the unit step function.
(b) Find the derivative $dx(t)/dt$.

1.7 Express the signals in Figure 1.64 in terms of the unit step and unit ramp functions.

1.8 If
$$x(t) = \begin{cases} 0, & t < 0 \\ 8, & 0 < t < 2 \\ 2t + 6, & 2 < t < 6 \\ 0, & t > 6 \end{cases}$$
then express x(t) in terms of singularity functions.

1.9 Sketch the following signals:
(a) $x(t) = u(t + 2) - u(t - 2)$
(b) $y(t) = 4\Pi\,[(t + 1)/2]$
(c) $z(t) = \begin{cases} 2t + 1, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$

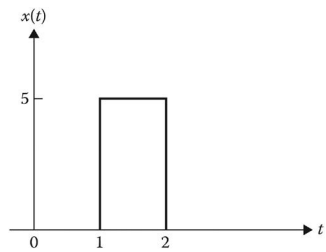


FIGURE 1.63 For Problem 1.6.

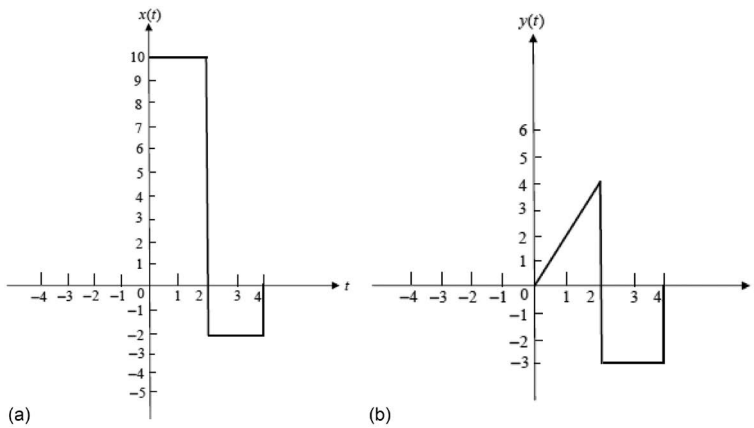


FIGURE 1.64 For Problem 1.7.

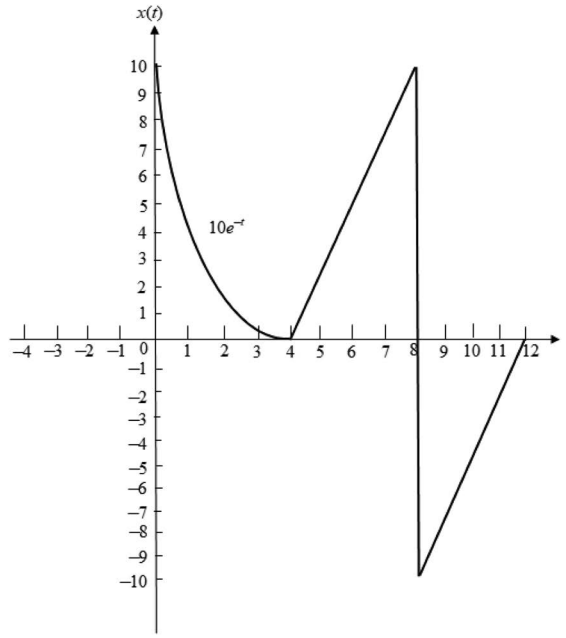


FIGURE 1.65 For Problem 1.10.

1.10 (a) Express $x(t)$ in Figure 1.65 in terms of the unit step function.
(b) Sketch the derivative of $x(t)$.

1.11 Let $$y(t) = \begin{cases} 2t, & 0 < t < 1 \\ -1, & 1 < t < 3 \\ 0, & \text{otherwise} \end{cases}$$

Express $y(t)$ in terms of singularity functions.

1.12 (a) Express the signal in Figure 1.66 in terms of the unit step function.
(b) Express $y(t)$ in terms of the rectangular pulse.

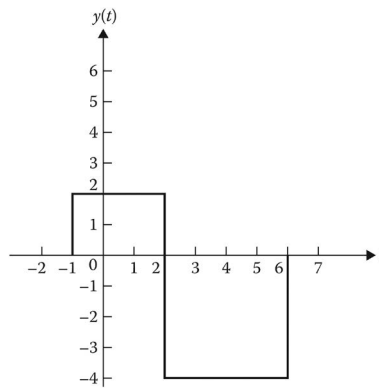


FIGURE 1.66 For Problem 1.12.

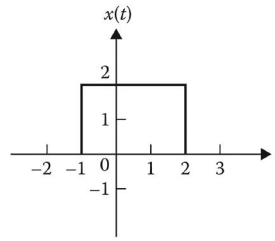


FIGURE 1.67 For Problem 1.13.

1.13 Given the signal $x(t)$ in Figure 1.67:
(a) Express $x(t)$ in terms of the unit step function.
(b) Express $x(t)$ in terms of the rectangular function.

1.14 Evaluate the following integrals:

(a) $\int_{-4}^{1}(t-1)\delta(t-0.5)dt$

(b) $\int_{-4}^{1}\exp(1-2t)\delta(t-2)dt$

1.15 Evaluate the following integral:

$$I=\int_{-\infty}^{\infty}\left[\sin\pi t\delta(t+0.5)+2e^{t-1}\delta(t)+\cos\pi t\delta(t-0.5)\right]dt$$

1.16 Evaluate the following integrals:

(a) $\int_{1}^{t}u(\lambda)d\lambda$

(b) $\int_{0}^{4}r(t-1)dt$

(c) $\int_{0}^{4}(t-6)^2\delta(t-2)dt$

(d) $\int_{-4}^{4}2t^3\delta(t-4)dt$

1.17 Evaluate the following derivatives:

(a) $\frac{d}{dt}\left[u(t-1)u(t+1)\right]$

(b) $\frac{d}{dt}\left[r(t-6)u(t-2)\right]$

(c) $\frac{d}{dt}\left[\sin 4tu(t-3)\right]$

1.18 Consider the triangular function:

$$x(t)=\begin{cases}\dfrac{t+a}{a}, & -a<t<0\\ \dfrac{a-t}{a}, & 0<t<a\\ 0, & \text{otherwise}\end{cases}$$

Show that $x''(t)=\dfrac{d^2x}{dt^2}=\dfrac{1}{a}\delta(t+a)-\dfrac{2}{a}\delta(t)+\dfrac{1}{a}\delta(t-a)$

1.19 Consider the signal

$$x(t)=\begin{cases}-t/2, & 0<t<1/2\\ \dfrac{t}{2}-\dfrac{1}{2}, & 1/2<t<1\\ 0, & \text{otherwise}\end{cases}$$

(a) Find the derivative $x'(t)$.
(b) Sketch $x(t)$ and $x'(t)$.

Section 1.5: Basic Discrete-Time Signals

1.20 Sketch the following signals:

(a) $x[n]=\begin{cases}n/2, & -3\le n\le 3\\ 0, & \text{otherwise}\end{cases}$

(b) $y[n]=(0.6)^n u[n], \quad 0\le n\le 4$

1.21 Sketch:

(a) $x[n]=\begin{cases}4(2)^n, & n\le -2\\ 0, & n<-2\end{cases}$

(b) $y[n]=\begin{cases}2(-2)^n, & n\le -2\\ 0, & n<-2\end{cases}$

1.22 Sketch the following discrete-time signals:

(a) $x[n]=\begin{cases}1, & -3\le n\le 3\\ 0, & \text{otherwise}\end{cases}$

(b) $$x(t) = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 2 \\ 2, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

(c) $x[n] = 2\sin(n/4)\cos(n\pi/8)$

1.23 Sketch the following signals:

(a) $x[n] = u[n - 2]$

(b) $y[n] = u[n + 2]$

(c) $z[n] = u[n] - u[n - 4]$

1.24 Sketch the following discrete-time signals:

(a) $x[n] = x[n] + u[-n]$

(b) $x[n] = 10\sin(2\pi n/20)$

1.25 Represent the signal in Figure 1.68 as a sum of impulses.

1.26 Sketch the signal:

$$x[n] = \begin{cases} 1, & -3 \le n < 0 \\ -2, & 1 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

1.27 Sketch:

(a) $x[n] = (2n + 1)u[n]$

(b) $$y[n] = \begin{cases} 1, & -2 \le n \le 2 \\ -2, & 3 \le n \le 5 \end{cases}$$

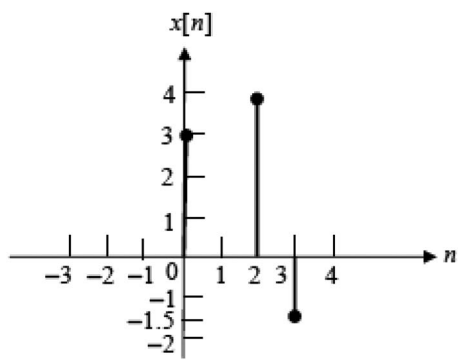


FIGURE 1.68 For Problem 1.25.

Section 1.6: Basic Operations on Continuous Signals

1.28 A signal $x(t)$ is shown in Figure 1.69. Sketch: (a) $x(2t)$ and (b) $x(-t)$.

1.29 Sketch the signal:

$$x(t) = \begin{cases} 2u(-t)+1, & -4 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

where $u(t)$ is a unit step function.

1.30 Given the signal $x(t)$ in Figure 1.70, sketch:
(a) $x(t-2)$ (b) $x(2-t)$

1.31 Given the signal $x(t)$ in Figure 1.71, sketch
(a) $x(t/3)$ (b) $x(t-2)$

1.32 (a) Express the signal $x(t)$ in Figure 1.72 in terms of the singularity function.
(b) Obtain the derivative of $x(t)$, i.e., $dx(t)/dt$.

1.33 (a) Express $y(t)$ in Figure 1.73 in terms of the triangular function.
(b) Find and sketch the derivative $y'(t)$ and $y''(t)$.

1.34 Consider the signal $x(t)$ in Figure 1.74. Sketch:
(a) $y(t) = x(t-1)$ (b) $z(t) = x(1+2t)$.

1.35 If $x(t)$ is the signal shown in Figure 1.75, sketch: (a) $x(t-2)$, (b) $x(3t)$, (c) $y(t) = 1 + 2x(t)$.

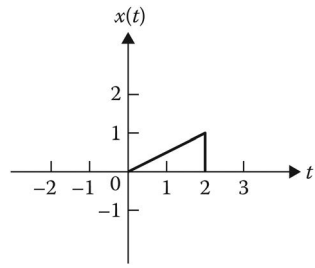


FIGURE 1.69 For Problem 1.28.

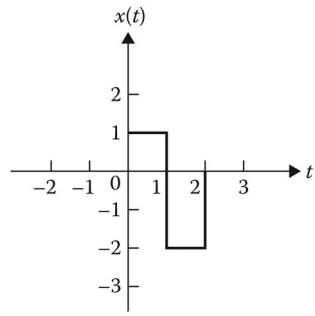


FIGURE 1.70 For Problem 1.30.

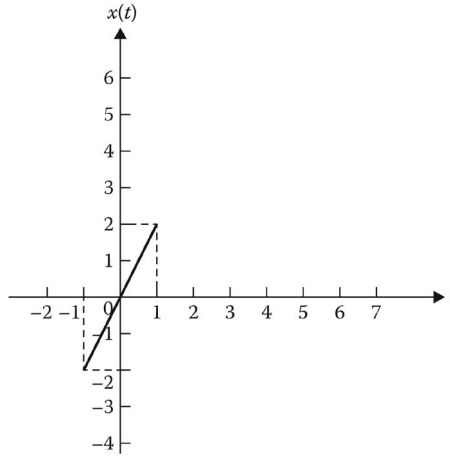


FIGURE 1.71 For Problem 1.31.

1.36 Sketch each of the following continuous-time signals and their derivatives:
(a) $x(t) = u(t) - u(t-1)$ and dx/dt
(b) $y(t) = u(t+1) - 2u(t) + u(t-1)$ and dy/dt
(c) $z(t) = t\,[u(t+1) - u(t-1)]$ and dz/dt

1.37 If $x(t) = \Pi(t+2) + \Pi(t-2)$, sketch:
(a) $y(t) = x(t-2)$
(b) $y(t) = x(t+1) - x(t-1)$

1.38 Sketch these signals:
(a) $u[(t-1)/2]$
(b) $r[4-2t]$
(c) $\prod(-2t+1)$
(d) $\Lambda[(t-1)/3]$

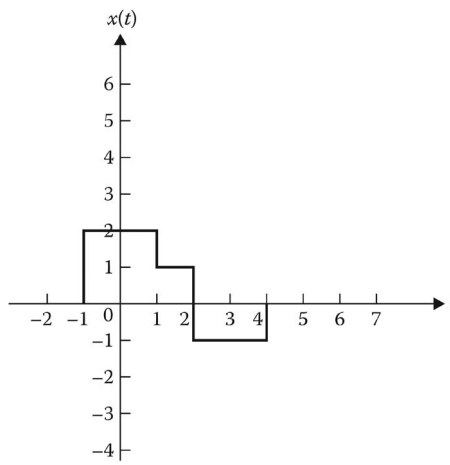


FIGURE 1.72 For Problem 1.32.

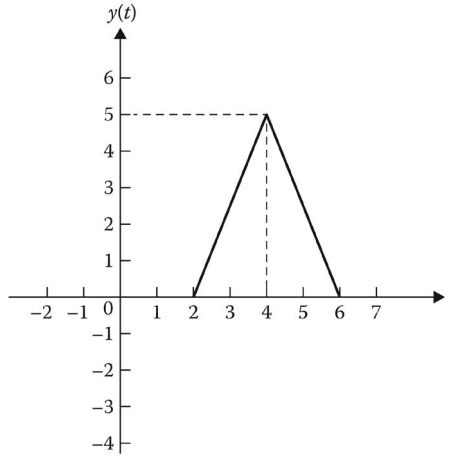


FIGURE 1.73 For Problem 1.33.

1.39 Given the two signals $x_1(t)$ and $x_2(t)$ in Figure 1.76, sketch the following:
(a) $y(t) = x_1(t) + x_2(t)$
(b) $z(t) = 0.5x_1(2t) + x_2(t)$
(c) $h(t) = x_1(t-1) - x_2(1-t)$

1.40 Given the signal in Figure 1.77, sketch: (a) $x(t + 2)$ and (b) $x(1 + 2t)$.

Section 1.7: Basic Operations on Discrete Signals

1.41 Given the discrete-time signal in Figure 1.78, sketch the following signals:
(a) $y[n] = x[n - 3]$
(b) $z[n] = x[n] - x[n - 1]$

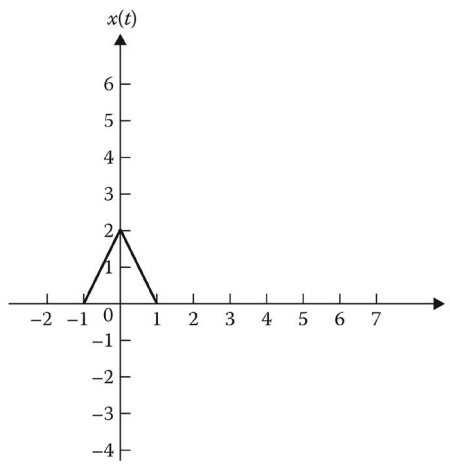


FIGURE 1.74 For Problem 1.34.

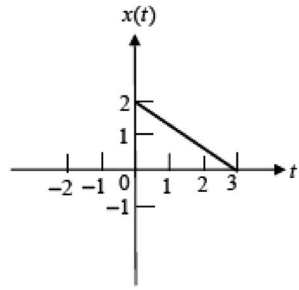


FIGURE 1.75 For Problem 1.35.

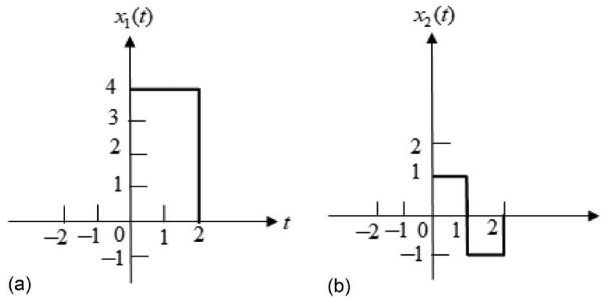


FIGURE 1.76 For Problem 1.39.

1.42 Consider the discrete-time signal in Figure 1.79. Sketch the following signals:
(a) $x[n]u[2-n]$
(b) $x[n][u[n+1]-u[n]]$
(c) $x[n]\delta[n-2]$

1.43 Given that $x[n] = \delta[n+1] - 2\delta n] + \delta[n-2]$,
(a) sketch $x[n]$, (b) sketch $x[n-1]$, (c) sketch $x[n+2]$.

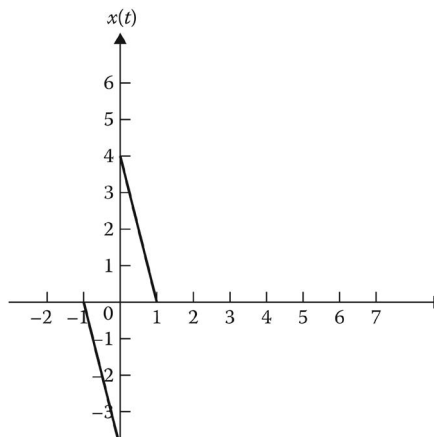


FIGURE 1.77 For Problem 1.40.

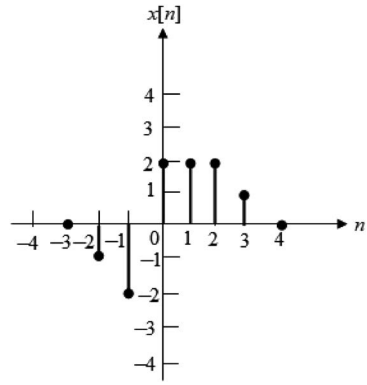


FIGURE 1.78 For Problem 1.41.

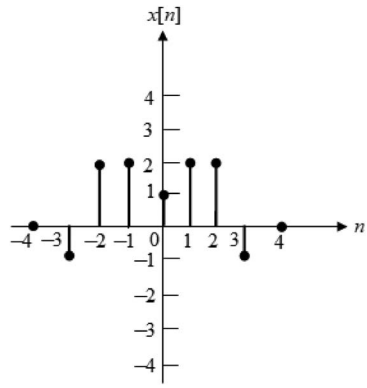


FIGURE 1.79 For Problem 1.42.

1.44 Consider $x[n]$ as shown Figure 1.80, sketch:

(a) $x[n-1]$
(b) $x[n+2]$
(c) $-3x[n-1]$
(d) $y[n] = u[n] + x[n-1]$

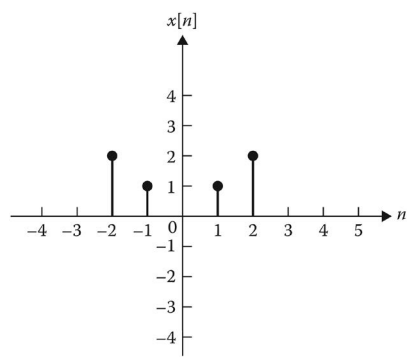


FIGURE 1.80 For Problem 1.44.

Section 1.8: Classification of Systems

1.45 Show that the system described by the differential equation

$$\frac{dy(t)}{dt} + 2ty(t) = x(t)$$

is linear.

1.46 Show that the system described by the differential equation

$$\frac{dy(t)}{dt} + 4y(t) + 1 = x(t)$$

is nonlinear.

1.47 Which of the following systems can be classified as linear/nonlinear, time-varying/time-invariant?

(a) $\dfrac{d^2y(t)}{dt^2} + 2ty(t) = f(t)$

(b) $\dfrac{d^2y(t)}{dt^2} + y(t)\dfrac{dy(t)}{dt} + y(t) = 4f(t)$

(c) $\dfrac{d^4y(t)}{dt^4} - 5\dfrac{d^3y(t)}{dt^3} + 2\dfrac{dy(t)}{dt} - y(t) = \dfrac{df(t)}{dt} - 4f(t)$

(d) $t^2\dfrac{d^2y(t)}{dt^2} + y^2(t) = f(t)$

1.48 Determine which of the following systems is linear:

(a) $y(t) = \exp[x(t)]$

(b) $y(t) = \cos x(t)$

(c) $y(t) = t^2x(t)$

1.49 Determine which of the following systems is linear or nonlinear:

(a) $\dfrac{dy}{dt} + y(t) = x^2(t)$

(b) $\left(\dfrac{dy}{dt}\right)^2 + y(t) = x(t)$

(c) $y(t) = \displaystyle\int_{-\infty}^{t} x(\lambda)d\lambda$

1.50 Determine whether the discrete-time system described by $y[n] = nx[n]$ is time-varying.

1.51 Determine whether the following systems are causal or noncausal, nonmemory or with memory.

(a) $y(t) = e^{x(t)}\sin t$

(b) $y(t) = \displaystyle\int_{0}^{t} x(\tau)\tau d\tau$

1.52 Consider the following integrator:

$$y(t) = \int_0^t x(\lambda)d\lambda$$

Check for time-invariance.

1.53 Show whether the following systems are time-invariant:
(a) $y(t) = tx(t) + 5$
(b) $y[n] = x^2[n]$

Section 1.9: Applications

1.54 The unit delay element is shown in Figure 1.81(a). Determine the input-output relation for the system in Figure 1.81(b).

1.55 Write the differential equation for $i(t)$ in the circuit of Figure 1.82. Assume that the capacitor is initially uncharged and that the switch closes at $t = 0$.

1.56 Write the system equation for the electric circuit of Figure 1.83.

1.57 Consider the RC circuit in Figure 1.84. Obtain the relationship between the input $x(t)$ and the output $y(t)$.

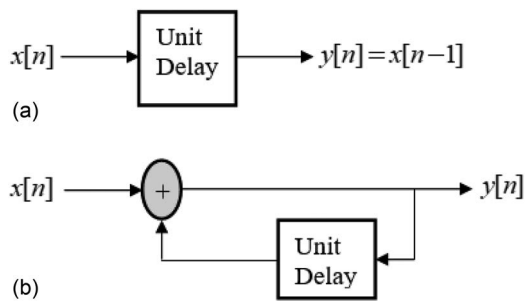


FIGURE 1.81 For Problem 1.54.

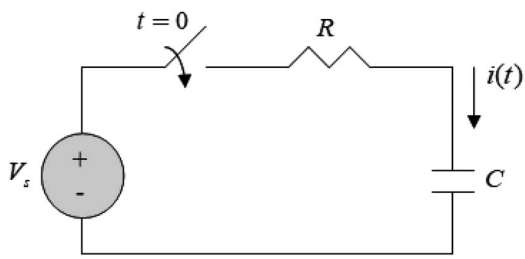


FIGURE 1.82 For Problem 1.55.

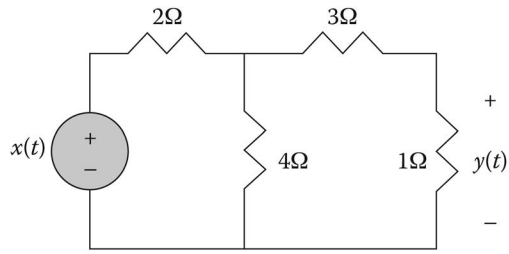


FIGURE 1.83 For Problem 1.56.

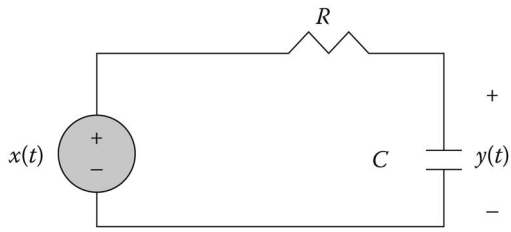


FIGURE 1.84 For Problem 1.57.

Section 1.10: Computing with MATLAB

1.58 Determine the numeric value of x in each of the following MATLAB instructions:
(a) t = pi/3; x = cos(t);
(b) t = 1:4; x = 2*sin(t);

1.59 A signal is given by

$$x(t) = \begin{cases} \cos(3000\pi t), & 0 < t < 10\text{ms} \\ 0, & \text{otherwise} \end{cases}$$

Use MATLAB to plot the signal.

1.60 Use MATLAB to plot the signal $y(t) = \begin{cases} 2t, & 0 < t < 1 \\ \frac{1}{2}(t^2 - 4t + 3), & 1 < t < 3 \\ 0, & \text{otherwise} \end{cases}$.

1.61 Use MATLAB to plot these discrete-time signals.
(a) $x[n] = 10(0.7)^n$, $n \geq 0$
(b) $y[n] = 10(1.2)^n$, $n \geq 0$

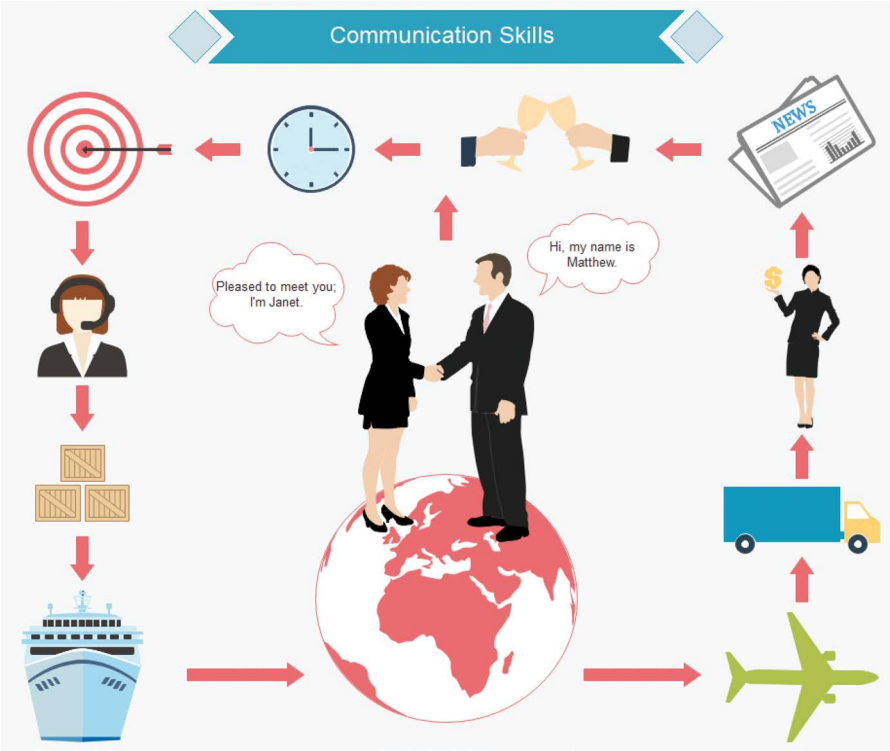
1.62 Use MATLAB to plot the following signals over $-2 \leq t \leq 4$ seconds.
(a) $x(t) = 2\ r(t)$
(b) $y(t) = 5e^{-2t}\ u(t)$
(c) $z(t) = 4\cos 4t + 2\ \sin(2t - \pi/4)$

2 Convolution

Humility allows us to experience victory without conceit and defeat without shame. Thus, humility is the key to constant learning and long-term success.

Anonymous

ENHANCING YOUR COMMUNICATION SKILLS



Taking a class in signals and systems is a means to preparing yourself for a career in electrical engineering. Enhancing your communication skills while in school should also be part of that preparation because, as an engineer, a large part of your time will be spent in communicating. No matter how you try, you cannot avoid communication. Virtually everyone communicates and interacts at work. Communication skills will be essential to your success as an engineer.

The Accreditation Board for Engineering and Technology (ABET) requires that graduates of engineering programs possess an ability to communicate effectively. It has been observed that graduating students are poorly prepared with respect to communication

DOI: 10.1201/9781003490869-2

skills. It has also been observed that the ability to communicate is the most important factor to promotion, more important than motivation, education, and hard work.

Communication is the process of transferring information from the sender and receiver through a medium. Oral, written, and graphical communication competence is an important professional skill for engineers. Professional engineers apply communication skills for many purposes, e.g., emailing messages, describing solutions, pitching ideas, reporting experiments, discussing work with collaborators or customers, presenting papers at conferences, etc.

Effective communication is a learned skill. It takes times and effort to develop these skills and become an effective communicator. Developing the skills while in school is highly recommended. Look for continuing opportunities to develop and strengthen your reading, writing, listening, and speaking skills. You can do this through classroom presentations, team projects, active participation in student organizations, and enrollment in communication courses. Learn to write good laboratory reports in terms of grammar, spelling, accuracy, and coherence. The risks are less here than in the workplace. Effective communication can improve relationships at home, at the workplace, and in social situations.

2.1 INTRODUCTION

As we saw in Chapter 1, a system may be regarded as a process that transforms an input signal to an output signal. The system must be capable of accepting input signals, operating on them, and producing output signals. The behavior of the system can be described mathematically either in the time domain, as is done in this chapter, or in the frequency domain, as in later chapters.

In this chapter, we will seek to understand a technique called *convolution*, which is a tool for time-domain analysis of systems. We will learn how to apply it in finding the response of linear, time-invariant (LTI) systems to input signals. The reason we limit our discussion to LTI systems is twofold. First, several physical systems can be modeled as LTI systems. Second, no general procedures exist for non-LTI systems. LTI systems can be analyzed in great detail because standard procedures are already available.

We will begin with applying convolution to continuous-time systems; we later will apply it to discrete-time systems. We use MATLAB® for evaluating the convolution of two signals. We will finally show how convolution can be applied in determining system stability and analyzing electric circuits.

2.2 IMPULSE RESPONSE

We recall from Chapter 1 that if $x(t)$ is the input signal and $y(t)$ is the output signal or response of a system; they are related through a transformation

$$y(t) = \boldsymbol{T}\ x(t) \tag{2.1}$$

where $\boldsymbol{T}$ is an operator transforming $x(t)$ into $y(t)$. The *impulse response* $h(t)$ is the response of the system when the input is the unit impulse function $\delta(t)$; i.e.,

$$h(t) = \boldsymbol{T}\ \delta(t) \tag{2.2}$$

The **impulse response** to an LTI system is the output of the system to a unit impulse function.

From Table 1.1, the input $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \tag{2.3}$$

where τ is a dummy variable. Equation (2.3) is the shifting property of the unit impulse.

The response $y(t)$ to the input $x(t)$ is obtained by combining eqs. (2.1) and (2.3).

$$\begin{aligned} y(t) = \mathbf{T}x(t) &= \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\left\{\delta(t-\tau)\right\}d\tau \end{aligned} \tag{2.4}$$

in which the interchange of operations is allowed by virtue of the system's linearity. Since we are assuming that the system is time-invariant, eq. (2.2) implies that

$$h(t-\tau) = \mathbf{T}\left\{\delta(t-\tau)\right\} \tag{2.5}$$

From eqs. (2.4) and (2.5), we get

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2.6}$$

This shows that an LTI system is characterized by its impulse response.

2.3 THE CONVOLUTION INTEGRAL

Equation (2.6) is known as the *convolution integral* or *superposition integral*. The convolution integral occurs frequently in science, mathematics, and engineering. The convolution of two signals $x(t)$ and $h(t)$ is usually written in terms of the operator * as

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2.7}$$

that is, $y(t)$ equals $x(t)$ convolved with $h(t)$. The asterisk denotes convolution in this chapter and should not be confused with a complex conjugate. A block diagram illustration of convolution is given in Figure 2.1.

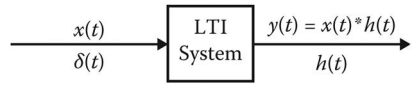


FIGURE 2.1 An illustration of convolution in an LTI system.

We can split the integral in eq. (2.7) into two parts.

$$y(t) = x(t) * h(t) = \underbrace{\int_{-\infty}^{t_o} x(\tau)h(t-\tau)d\tau}_{y_{zir}} + \underbrace{\int_{t_o}^{\infty} x(\tau)h(t-\tau)d\tau}_{y_{zsr}} = y_{zir}(t) + y_{zsr}(t) \tag{2.8}$$

where y_{zir} (t) is the zero-input response (or the natural response) of the system and $y_{zsr}(t)$ is the zero-state response (or the forced response) of the system and t_o is the initial time. Thus the complete response of a physical system is divided into the zero-input response and the zero-input response.

The convolution integral in eq. (2.7) is the general one. It applies to any LTI system. However, the convolution integral can be simplified if we assume that a system has two properties. First, if $x(t) = 0$ for $t < 0$, then

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2.9a}$$

Second, if we assume that the system is causal, $h(t) = 0$ for $t < 0$, then $h(t - \tau) = 0$ for $t - \tau < 0$ or $\tau > t$, so that eq. (2.9a) becomes

$$y(t) = x(t) * h(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau \tag{2.9b}$$

Some important properties of the convolution integral are listed in Table 2.1.

Property #1 states that the order in which two functions are convolved is unimportant. We will see shortly how to take advantage of this commutative property when performing graphical computation of the convolution integral.

Another important property of the convolution integral is the width property. If the durations of $x(t)$ and $h(t)$ are T_1 and T_2, respectively, then the duration of $y(t) = x(t)*h(t)$ is $T_1 + T_2$. If the areas under $x(t)$ and $h(t)$ are A_1 and A_2, respectively, then the area under $y(t) = x(t)*h(t)$ is A_1A_2, i.e.,

$$\text{Area under } y(t) = \text{Area under } x(t) \times \text{Area under } h(t) \tag{2.10}$$

Of course, this does not hold for signals extending to $\pm\infty$.

TABLE 2.1
Properties of the Convolution Integral

1. $x(t)*h(t) = h(t)*x(t)$ (Commutative)
2. $f(t)*[x(t) + y(t)] = f(t)*x(t) + f(t)*y(t)$ (Distributive)
3. $f(t)*[(x(t)*y(t)] = [f(t)*x(t)]*y(t)$ (Associative)
4. $f(t)*\delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$
5. $f(t)*\delta(t-t_o) = f(t-t_o)$
6. $f(t)*\delta'(t) = \int_{-\infty}^{\infty} f(\tau)\delta'(t-\tau)d\tau = f'(t)$
7. $f(t)*u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau$
8. $u(t)*\delta'(t) = \int_{-\infty}^{\infty} u(\tau)\delta'(t-\tau)d\tau = u'(t) = \delta(t)$
9. If $x_1(t)*x_2(t) = y(t)$, then
   $x_1(t+t_1)*x_2(t+t_2) = y(t+t_1+t_2)$

We can also add that if

$$x_1(t)*x_2(t) = y(t)$$

then

$$x_1(t+t_1)*x_2(t) = y(t+t_1) \tag{2.11a}$$

$$x_1(t+t_1)*x_2(t+t_2) = y(t+t_1+t_2) \tag{2.11b}$$

$$y(at) = a\ x_1(at)*x_2(at), \quad a>0 \text{ (time scaling)} \tag{2.11c}$$

$$y(-t) = x_1(-t)*x_2(-t) \text{ (time reversal)} \tag{2.11d}$$

The convolution integral can be evaluated in three different ways:

1. Analytical method, which involves performing the integration by hand when $x(t)$ and $h(t)$ are specified analytically.
2. Graphical method, which is appropriate when $x(t)$ and $h(t)$ are provided in graphical form.
3. Numerical method, where we approximate $x(t)$ and $h(t)$ by numerical sequence and obtain $y(t)$ by discrete convolution using a digital computer.

We will apply the analytical method right away and apply graphical and numerical methods later. Table 2.2 shows some common signals and their convolutions.

TABLE 2.2
An Abbreviated Convolution Table

No.	$x_1(t)$	$x_2(t)$	$x_1(t)*x_2(t)$
1.	$x(t)$	$\delta(t)$	$x(t)$
2.	$u(t)$	$u(t)$	$tu(t) = r(t)$
3.	$e^{at}u(t)$	$u(t)$	$\frac{e^{at}-1}{a}u(t)$
4.	$e^{-at}u(t)$	$e^{-bt}u(t)$	$\frac{1}{a-b}[e^{-bt}-e^{-at}]u(t)$
5.	$e^{at}u(t)$	$e^{bt}u(t)$	$\frac{1}{b-a}[e^{bt}-e^{at}]u(t)$
6.	$e^{at}u(t)$	$e^{at}u(t)$	$te^{at}u(t)$
7.	$te^{at}u(t)$	$e^{at}u(t)$	$\frac{1}{2}t^2e^{at}u(t)$

Example 2.1

Obtain the convolution of these two signals:

$$x(t) = 4e^{-t}u(t) \quad \text{and} \quad h(t) = e^{-3t}u(t)$$

SOLUTION

Method 1: Let $y(t) = x(t)*h(t)$. Applying eq. (2.6), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} 4e^{-\tau}u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau \tag{2.1.1}$$

Consider $u(\tau)$ and $u(t-\tau)$ as shown in Figures 2.2(a) and (b). It is evident that the two functions overlap on $0 < \tau < t$. Within that interval, their product is 1. Outside the interval, their product is 0. Their product is shown in Figure 2.2(c). Hence the limit $-\infty < \tau < \infty$ becomes $0 < \tau < t$. Thus, eq. (2.1.1) becomes

$$\begin{aligned} y(t) &= \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t 4e^{-\tau}(1)e^{-3(t-\tau)}(1)d\tau = 4e^{-3t}\int_0^t e^{-\tau}e^{3\tau}\,d\tau \\ &= 4e^{-3t}\left(\frac{e^{2\tau}}{2}\right)\Bigg|_0^t = 2e^{-3t}(e^{2t}-1) = 2(e^{-t}-e^{-3t})u(t) \end{aligned} \tag{2.1.2}$$

Method 2: Let $y(t) = h(t)*x(t)$. Then,

$$\begin{aligned} y(t) &= \int_0^t h(\tau)x(t-\tau)d\tau = \int_0^t e^{-3\tau}(1)4e^{-(t-\tau)}(1)d\tau = 4e^{-t}\int_0^t e^{-3\tau}e^{\tau}\,d\tau \\ &= 4e^{-t}\left(\frac{e^{-2\tau}}{-2}\right)\Bigg|_0^t = 2e^{-t}(1-e^{-2t}) = 2(e^{-t}-e^{-3t})u(t) \end{aligned} \tag{2.1.3}$$

which agrees with eq. (2.1.2). The trick is that when convolving two functions, fold the easier function.

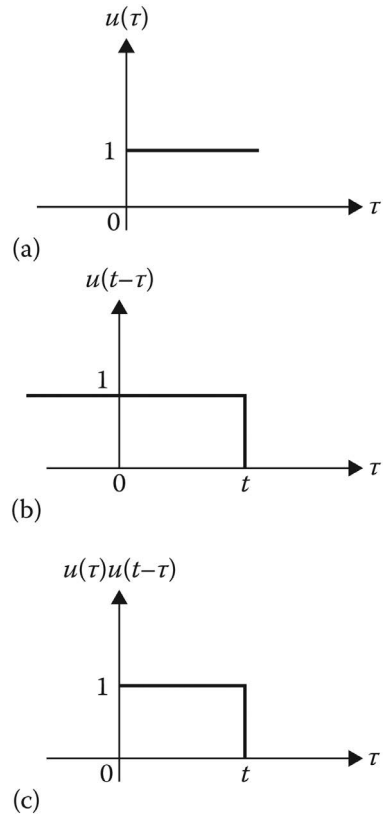


FIGURE 2.2 Multiplying $u(\tau)$ and $u(t-\tau)$.

Practice Problem 2.1 Determine the convolution of $x(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}\ u(t)$

Answer: $e^{-t} - e^{-2t}, t > 0$

Example 2.2

The input $x(t)$ and the impulse response $h(t)$ of an LTI system are given by $x(t) = u(t)$ and $h(t) = e^{-3t}u(t)$. Find the output response.

SOLUTION

From eq. (2.6),

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)e^{-3(t-\tau)}\ u(t-\tau)d\tau \tag{2.2.1}$$

In this integral, t is a constant so that we can pull out the factor e^{-3t}. What we explained in Figure 2.2 for Example 2.1 can also be explained as follows. By definition,

$$u(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

$$u(t-\tau) = \begin{cases} 1, & t-\tau > 0 \\ 0, & t-\tau < 0 \end{cases} = \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases}$$

Thus,

$$u(\tau)u(t-\tau) = \begin{cases} 1, & 0 < t < \tau \\ 0, & \text{otherwise} \end{cases}$$

The limits on the integral now become $0 < \tau < t$. Therefore, eq. (2.2.1) becomes

$$y(t) = e^{-3t} \int_0^t e^{3\tau}\, d\tau = \frac{e^{-3t}}{3} e^{3\tau} \Big|_0^t$$

$$= \frac{1}{3}(1 - e^{-3t}), \quad t > 0$$

or

$$y(t) = \frac{1}{3}(1 - e^{-3t})u(t)$$

Practice Problem 2.2 Consider a system in which the input $x(t) = r(t) = tu(t)$ and the impulse response $h(t) = u(t)$. Determine the output $y(t)$.

Answer: $y(t) = 0.5t^2u(t)$.

2.4 GRAPHICAL CONVOLUTION

We now consider the graphical method of evaluating the convolution integral. This method usually involves four steps:

1. Folding: Take the mirror image of $h(\tau)$ about the ordinate (or vertical) axis to obtain $h(-\tau)$.
2. Shifting: Displace or shift $h(-\tau)$ by t to obtain $h(t - \tau)$.
3. Multiplication: Multiply $h(t - \tau)$ and $x(\tau)$ together.
4. Integration: For a given t, integrate the product $h(t - \tau)\, x(\tau)$ over $0 < \tau < t$ to get $y(t)$ at t.

The folding operation in step 1 is the reason for the term *convolution*. The function $h(t - \tau)$ scans or slides over $x(\tau)$.

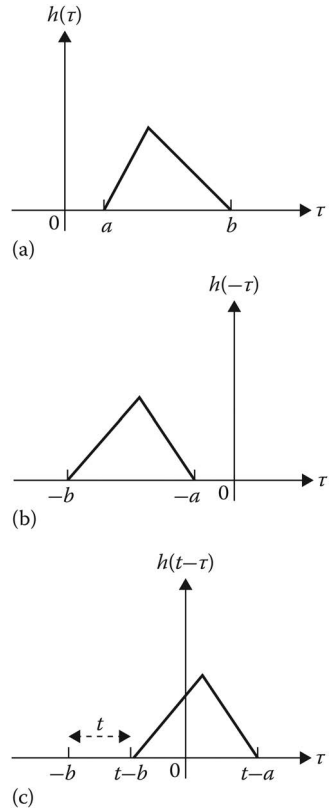


FIGURE 2.3 Folding and sliding the function $h(t)$.

To apply this procedure, we must be able to sketch $h(t - \tau)$. Sketching $h(t - \tau)$ is the key to the convolution process. It involves reflecting $h(\tau)$ about the vertical axis and shifting by t. This is illustrated in Figure 2.3. Analytically, we obtain $h(t - \tau)$ by replacing every t in $h(t)$ by $t - \tau$. Since convolution is commutative, it may be more convenient to apply steps 1 and 2 to $x(t)$ instead of $h(t)$.

Example 2.3

Obtain the convolution of the two signals in Figure 2.4.

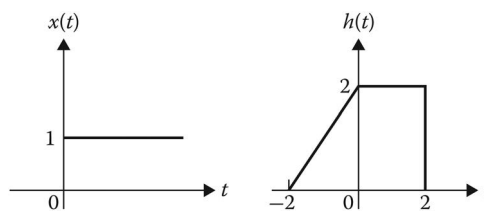


FIGURE 2.4 For Example 2.3.

SOLUTION

Let $y(t) = x(t)*h(t)$, where $x(t)$ is a unit step and

$$h(t) = \begin{cases} 2+t, & -2 < t < 0 \\ 2, & 0 < t < 2 \end{cases}$$

In this case, it is easy to fold $x(t)$, the unit step function. Let

$$y(t) = x(t) * h(t) = \int x(t-\tau)h(\tau)d\tau$$

We follow the four steps to get $y(t)$. First, we fold $x(t)$ as shown in Figure 2.5(a) and shift it by t as shown in Figure 2.5(b).

For $t < -2$, there is no overlap of the two signals, as shown in Figure 2.6(a). Hence,

$$y(t) = x(t) * h(t) = 0, \quad t < -2 \tag{2.3.1}$$

For $-2 < t < 0$, the two signals overlap between -2 and t, as shown in Figure 2.6(b).

$$\begin{aligned} y(t) &= \int_{-2}^{t} h(\tau)x(t-\tau)d\tau = (2+\tau)(1)d\tau \Big|_{-2}^{t} = 2\tau + \frac{\tau^2}{2}\Big|_{-2}^{t} \\ &= 0.5t^2 + 2t + 2, \quad -2 < t < 0 \end{aligned} \tag{2.3.2}$$

For $0 < t < 2$, the two signals overlap between -2 and t, as shown in Figure 2.6(c).

$$\begin{aligned} y(t) &= \int_{-2}^{0} (2+\tau)(1)d\tau + \int_{0}^{t} (2)(1)d\tau \\ &= \left(2\tau + \frac{\tau}{2}\right)\Big|_{-2}^{0} + 2\tau\Big|_{0}^{t} = 4 - 2 + 2t = 2(t+1), \quad 0 < t < 2 \end{aligned} \tag{2.3.3}$$

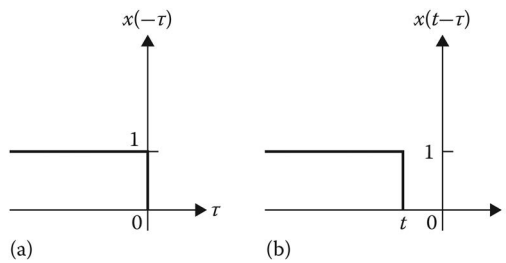


FIGURE 2.5 For Example 2.3: (a) Folding $x(t)$, (b) shifting $x(-\tau)$ by t.

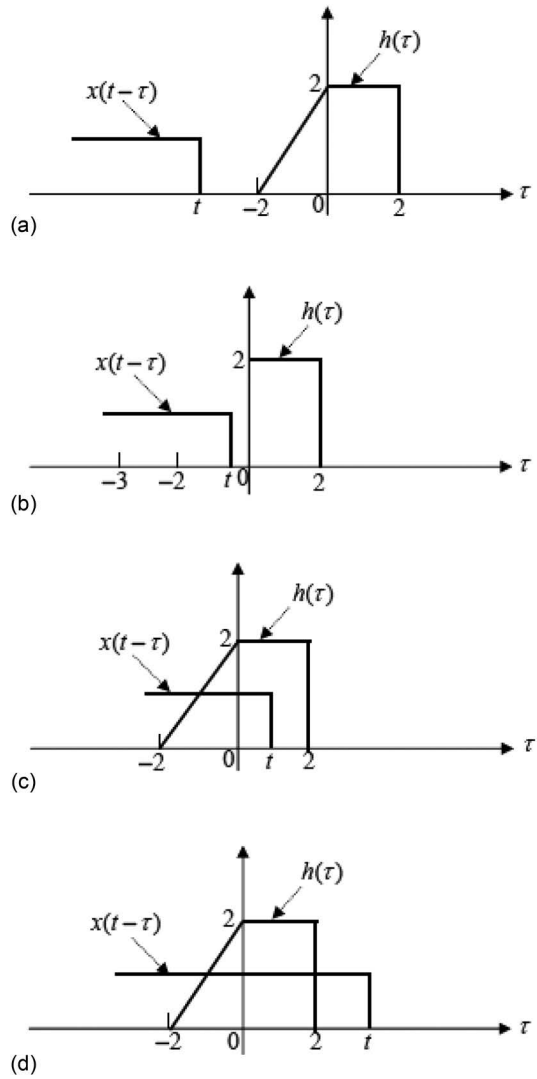


FIGURE 2.6 For Example 2.3: Overlapping of $x(t-\tau)$ and $h(\tau)$ for: (a) $t<-2$, (b) $-2<t<0$, (c) $0<t<2$, (d) $t>2$.

For $t>2$, the two signals overlap between –2 and 2, as shown in Figure 2.6(d).

$$\begin{aligned} y(t) &= \int_{-2}^{0}(2+\tau)(1)d\tau + \int_{0}^{2}(2)(1)d\tau \\ &= \left(2\tau+\frac{\tau}{2}\right)\Big|_{-2}^{0} + 2\tau\Big|_{0}^{2} = 4-2+4=6, \quad t>2 \end{aligned} \tag{2.3.4}$$

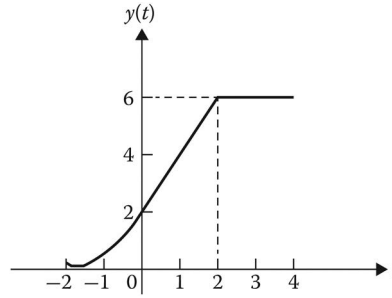


FIGURE 2.7 For Example 2.3: Convolution of signals $x(t)$ and $h(t)$.

Combining the results in eqs. (2.3.1) to (2.3.4), we obtain

$$y(t) = \begin{cases} 0.5t^2 + 2t + 2, & -2 \le t \le 0 \\ 2(t+1), & 0 \le t \le 2 \\ 6, & t > 2 \\ 0, & \text{otherwise} \end{cases} \tag{2.3.5}$$

which is sketched in Figure 2.7. Notice that $y(t)$ in this equation is continuous; i.e., it has no discontinuities. This fact can be used to check the results as we move from one range of t to another.

Practice Problem 2.3 Find the convolution of the two signals in Figure 2.8.

Answer:

$$z(t) = x(t) * y(t) = \begin{cases} \frac{1}{2}t^2, & 0 \le t \le 1 \\ \frac{1}{2}(2t-1), & 1 < t < 2 \\ \frac{1}{2}(3+2t-t^2), & 2 \le t \le 3 \\ 0, & \text{otherwise} \end{cases}$$

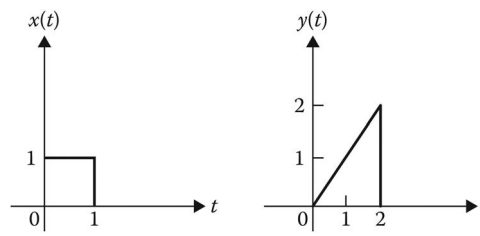


FIGURE 2.8 For Practice Problem 2.3.

Example 2.4

Given $x_1(t)$ and $x_2(t)$ in Figure 2.9, find $y(t) = x_1(t) * x_2(t)$.

SOLUTION

We fold $x_1(t)$ so that

$$y(t) = x_1(t) * x_2(t) = \int x_1(t-\tau)x_2(\tau)d\tau$$

For $t < -1$, there is no overlapping, as shown in Figure 2.10(a). Hence, $y(t) = 0$.
For $-1 < t < 0$, there is overlapping between -1 and t, as in Figure 2.10(b),

$$y(t) = \int_{-1}^{t} (t-\tau)^2(1)d\tau = \frac{(t-\tau)^3}{-3}\bigg|_{-1}^{t} = \frac{(t+1)^3}{3}, \quad -1 < t < 0$$

For $0 < t < 1$, the two signals overlap between $t-1$ and t, as shown in Figure 2.10(c).

$$y(t) = \int_{t-1}^{t} (t-\tau)^2(1)d\tau = \frac{(t-\tau)^3}{-3}\bigg|_{t-1}^{t} = \frac{1}{3}, \quad 0 < t < 1$$

For $1 < t < 2$, they overlap between $t-1$ and 1, as depicted in Figure 2.10(d).

$$y(t) = \int_{t-1}^{t} (t-\tau)^2(1)d\tau = \frac{(t-\tau)^3}{-3}\bigg|_{t-1}^{1} = \frac{1}{3} - \frac{1}{3}(t-1)^3, \quad 1 < t < 2$$

For $t > 2$, there is no overlapping so that $y(t) = 0$. Thus,

$$y(t) = \begin{cases} \frac{1}{3}(t+1)^3, & -1 < t < 0 \\ \frac{1}{3}, & 0 < t < 1 \\ \frac{1}{3} - \frac{1}{3}(t-1)^3, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

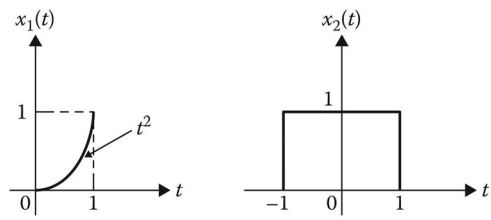


FIGURE 2.9 For Example 2.4.

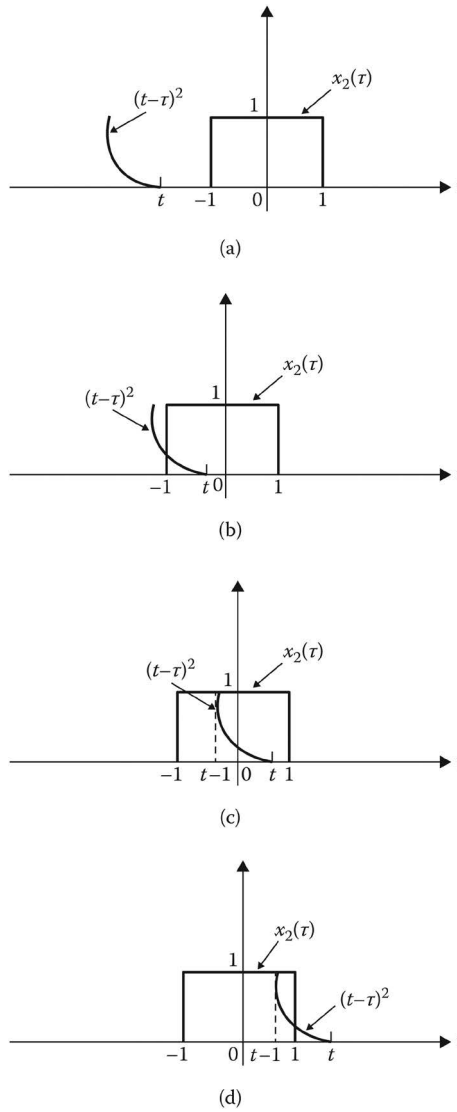


FIGURE 2.10 For Example 2.4: (a) $t < -1$, (b) $-1 < t < 0$, (c) $0 < t < 1$, (d) $t > 1$.

Practice Problem 2.4 Graphically convolve $x(t)$ and $y(t)$ shown in Figure 2.11.

Answer: $z(t) = x(t) * y(t) = \begin{cases} \frac{1}{\pi}(1-\cos \pi t), & 0 < t < 1 \\ \frac{1}{\pi}(\cos \pi t - 1), & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$

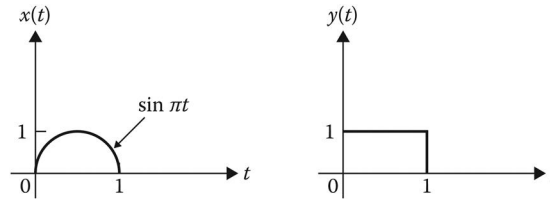


FIGURE 2.11 For Practice Problem 2.4.

2.5 BLOCK DIAGRAM REPRESENTATION

The commutative, associative, and distributive properties of the convolution integral, listed in Table 2.1, can be illustrated in the block diagrams shown in Figure 2.12. Notice that the commutative property, illustrated in Figure 2.12(a), shows that $x(t)$ and $h(t)$ are interchangeable. The associative property, illustrated in Figure 2.12(b), states that a series or cascaded arrangement of LTI systems can be replaced by a single system using convolution. The distributive property, illustrated in Figure 2.12(c), indicates that a parallel arrangement of LTI systems can be combined into a single system through addition.

Example 2.5

Find the impulse response of the system shown in Figure 2.13. Let

$$h_1(t) = 3\delta(t)$$
$$h_2(t) = 2e^{-t}u(t)$$
$$h_3(t) = 4e^{-2t}u(t)$$
$$h_4(t) = e^{-3t}u(t)$$

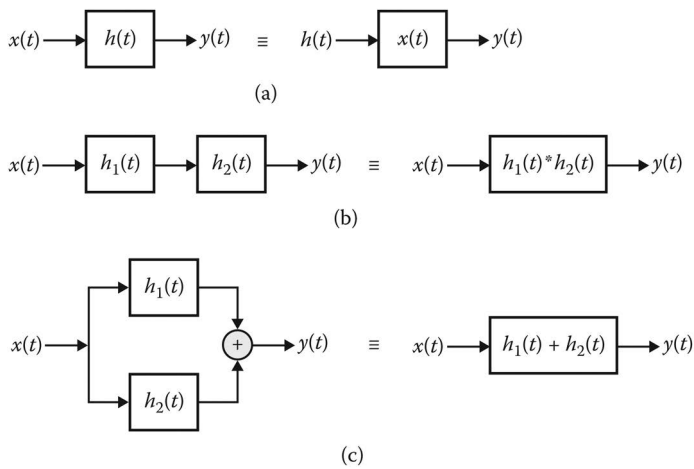


FIGURE 2.12 Block diagrams illustrating the properties of convolution.

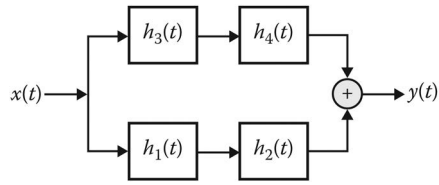


FIGURE 2.13 For Example 2.5.

SOLUTION

Comparing Figure 2.13 with Figure 2.12, we notice that

$$h(t) = h_1(t) * h_2(t) + h_3(t) * h_4(t) \tag{2.5.1}$$

Using the fact that $x(t)*\delta(t) = x(t)$, we obtain the first convolution on the right-hand side as

$$h_1(t) * h_2(t) = 3\delta(t) * \left\{2e^{-t}u(t)\right\} = 6e^{-t}u(t) \tag{2.5.2}$$

For the second convolution,

$$h_3(t) * h_4(t) = \int_{-\infty}^{\infty} 4e^{-2\tau}u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau$$

But

$$u(\tau)u(t-\tau) = \begin{cases} 1, & 0 < t < \tau \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} h_3(t) * h_4(t) &= \int_0^t 4e^{-2\tau}e^{-3(t-\tau)}d\tau = 4e^{-3t}\int_0^t e^{(3-2)\tau}d\tau \\ &= 4e^{-3t}e^{\tau}\Big|_0^t = 4(e^{-2t} - e^{-3t}), \quad t > 0 \\ &= 4(e^{-2t} - e^{-3t})u(t) \end{aligned} \tag{2.5.3}$$

Substituting eqs. (2.5.2) and (2.5.3) into eq. (2.5.1) gives

$$h(t) = 6e^{-t}u(t) + 4(e^{-2t} - e^{-3t})u(t)$$

Practice Problem 2.5 For the system shown in Figure 2.14,

$$h_1(t) = 2e^{-t}u(t)$$
$$h_2(t) = 4e^{-2t}u(t)$$

Determine the overall impulse response $h(t)$.

Answer: $h(t) = 8\left(e^{-t} - e^{-2t}\right)u(t)$.

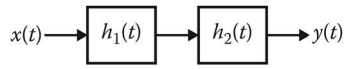


FIGURE 2.14 For Practice Problem 2.5.

2.6 DISCRETE-TIME CONVOLUTION

When an input is applied to a discrete-time system, the response or output sequence can be determined in a way similar to using the impulse response and the convolution integral for continuous-time systems. To refresh our minds, we will redefine the unit step sequence $u[n]$ as

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \tag{2.12}$$

The unit impulse sequence is also redefined as

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{2.13}$$

The signal only has value at $n = 0$. The displaced delta function is

$$\delta[n-k] = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases} \tag{2.14}$$

We said in Chapter 1 that an alternative way of expressing any discrete signal $x[n]$ is

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{2.15}$$

that is, we can represent $x[n]$ as a weighted sum of delayed impulses. The multiplication property of the impulse function is

$$\delta[n]x[n-k] = x[-k]\delta[n] \tag{2.16a}$$

$$\delta[n-k]x[n] = x[k]\delta[n-k] \tag{2.16b}$$

The impulse response $h[n]$ of a discrete-time LTI system is the response of the system when the input is $\delta[t]$, i.e.,

$$h[n] = \boldsymbol{T}\{\delta[t]\} \tag{2.17}$$

This is illustrated in Figure 2.15.

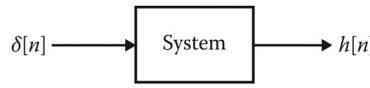


FIGURE 2.15 Block diagram illustrating the impulse response.

The convolution of the discrete input signal $x[n]$ and the impulse response $h[n]$ is

$$y[n] = x[n] * h[n] \tag{2.18}$$

and is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.19}$$

Thus,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.20}$$

This is known as the *convolution sum* or *superposition sum* for the system response. Notice that as in continuous-time convolution, one of the signals is time-inverted, shifted, and then multiplied by the other. By the change of variables $m = n - k$ or $k = n - m$, we have

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n] \tag{2.21}$$

This shows that the order of summation is immaterial; i.e., discrete convolution is commutative. This and other properties of the convolution operation are listed in Table 2.3. It is evident from the convolution sum in eq. (2.18) that if we know $h[n]$, we can find the system response $y[n]$ to any input $x[n]$.

TABLE 2.3
Properties of the Convolution Sum

1. $x[n]*h[n] = h[n]*x[n]$ (Commutative)
2. $f[n]*[x[n] + y[n]] = f[n]*x[n] + f[n]*y[n]$ (Distributive)
3. $f[n]*[(x[n]*y[n]] = [f[n]*x[n]]*y[n]$ (Associative)
4. $x[n - m]*h[n - k] = y[n - m - k]$ (Shifting)
5. $x[n]*\delta[n] = x[n]$

TABLE 2.4
An Abbreviated Table for Convolution Sums

No.	$x_1[n]$	$x_2[n]$	$x_1[n]*x_2[n]$
1.	$x[n]$	$\delta[n-k]$	$x[n-k]$
2.	$u[n]$	$u[n]$	$(n+1)u[n] = r[n]$
3.	$a^n u[n]$	$u[n]$	$\left[\frac{1-a^{n+1}}{1-a}\right]u[n]$
4.	$a^n u[n]$	$b^n u[n]$	$\left[\frac{a^{n+1}-b^{n+1}}{a-b}\right]u[n]$
5.	$u[n]$	$nu[n]$	$\frac{1}{2}(n)(n+2)u[n]$
6.	$nu[n]$	$nu[n]$	$1/6\, n(n+1)(n-1)u[n]$
7.	$a^n u[n]$	$nu[n]$	$\left[\frac{a(a^n-1)+n(1-a)}{(1-a)^2}\right]u[n]$
8.	$a^n u[n]$	$a^n u[n]$	$(n+1)a^n u[n]$

If both $x[n]$ and $h[n]$ are causal, i.e., $x[n]$ and $h[n]$ are zero for all integers $n < 0$, the summation in eq. (2.20) becomes

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k], \quad n \geq 0 \tag{2.22}$$

where $y[n] = 0$ for $n < 0$.

The convolution of an M-point sequence with an N-point sequence produces an $(M + N - 1)$-point sequence.

Just as the continuous-time convolution involves some steps, evaluating the convolution sum requires that the following steps:

1. The signal $h[k]$ is time-reversed to get $h[-k]$ and then shifted by n to form $h[n - k]$ or $h[-(k - n)]$, which should be regarded as a function of k with parameter n.
2. For a fixed value of n, multiply $x[k]$ and $h[n - k]$ for all values of k.
3. The product $x[k]h[n - k]$ is summed over all k to produce a single value of $y[n]$.
4. Repeat steps 1–3 for various values of n to produce the entire output $y[n]$.

Table 2.4 presents a list from which convolution sums can be determined directly for a variety of signal pairs.

Example 2.6

Let $r[n]$ be the convolution of two unit step sequences, i.e.,

$$r[n] = u[n] * u[n]$$

Find $r[n]$.

SOLUTION

$$r[n] = u[n] * u[n] = \sum_{k=-\infty}^{\infty} u[k]u[n-k]$$

The functions $u[k]$, $u[-k]$, and $u[n-k]$ are shown in Figure 2.16. The convolution takes place when we multiply the sequences in Figure 2.16(a) with Figures 2.16(c)

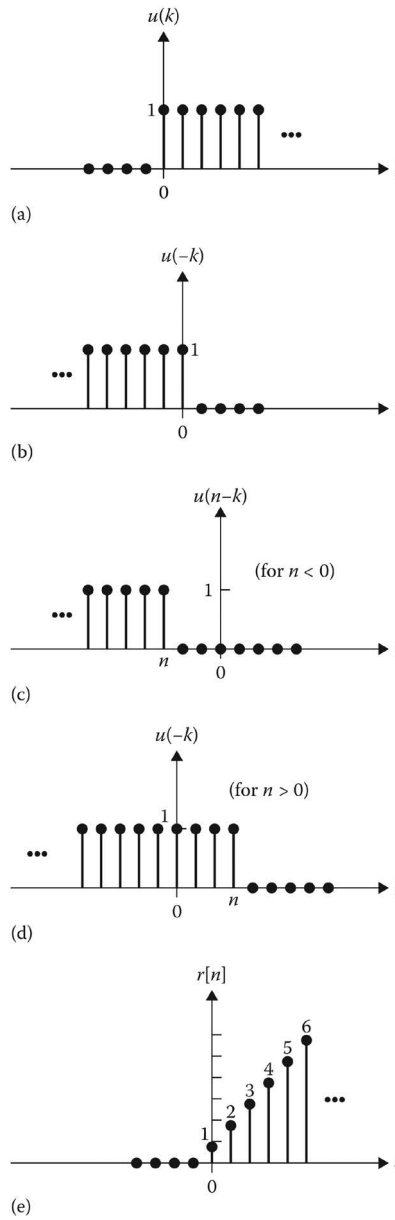


FIGURE 2.16 For Example 2.6.

and 2.16(d). It is good to show $u[n-k]$ for $n<0$ and for $n>0$ as in Figures 2.16(c) and 2.16(d), respectively. For $n<0$, the nonzero values of $u[k]$ and $u[n-k]$ do not overlap so that $u[k]u[n-k]=0$ for all values of k. This implies that $r[n]=0$ for $n<0$. For $n\geq 0$, the nonzero values of $u[k]$ and $u[n-k]$ overlap. This overlap begins with $u[k]$ at $k=0$ and ends with $u[n-k]$ at $k=n$. Hence

$$r[n]=\sum_{k=0}^{n}u[k]u[n-k]=\sum_{k=0}^{n}(1)=n+1$$

Therefore,

$$r[n]=(n+1)u[n]$$

This is the unit ramp sequence and is shown in Figure 2.16(e).

Practice Problem 2.6 Let the input $x[n]$ and impulse response of a system be given by

$$x[n]=u[n],\quad h[n]=a^{n}u[n],\quad 0<a<1$$

Find the output $y[n]=x[n]*h[n]$.

Answer: $y[n]=\dfrac{1-a^{n+1}}{1-a}u[n]$.

Example 2.7

Consider $x[n]$ and $h[n]$ as shown in Figure 2.17. (The signals are all zero outside the ranges indicated.) Find $y[n]=x[n]*h[n]$:

a. analytically, b. graphically.

SOLUTION

a. Using eq. (2.15), we can write

$$\begin{aligned}
x[n]&=\delta[n-1]+\delta[n-2]+\delta[n-3]\\
h[n]&=\delta[n]+\delta[n-1]+\delta[n-2]\\
y[n]&=x[n]*h[n]=x[n]*\{\delta[n]+\delta[n-1]+\delta[n-2]\}\\
&=x[n]+x[n-1]+x[n-2]\\
&=\delta[n-1]+\delta[n-2]+\delta[n-3]\\
&\quad+\delta[n-2]+\delta[n-3]+\delta[n-4]\\
&\quad+\delta[n-3]+\delta[n-4]+\delta[n-5]\\
&=\delta[n-1]+2\delta[n-2]+3\delta[n-3]+2\delta[n-4]+\delta[n-5]
\end{aligned}$$

This is illustrated in Figure 2.18.

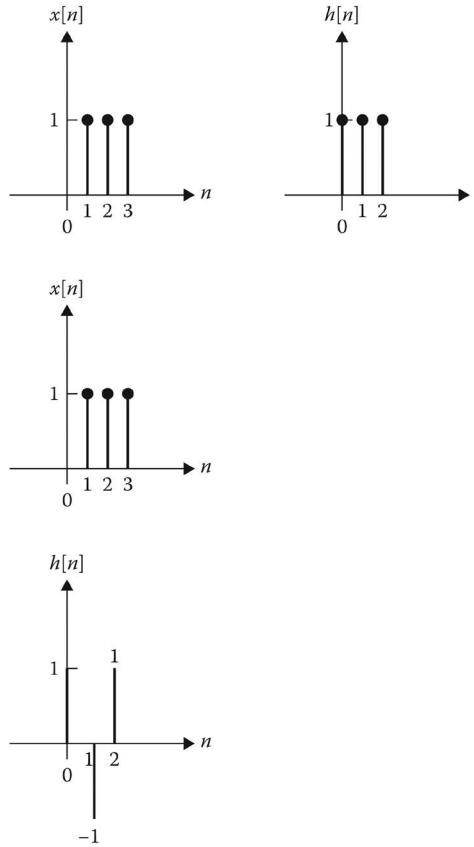


FIGURE 2.17 For Example 2.7.

b. The graphical convolution is shown in Figure 2.19. We do the convolution for $n = 0, 1, 2, 3$, and 4. Looking at the trend, we can similarly conclude that for $n = 5$, $y = 1$ and for $n > 5$, $y = 0$. Thus $y[n]$ is as shown in Figure 2.18, i.e.,

$$y[n] = [1, 2, 3, 2, 1]$$

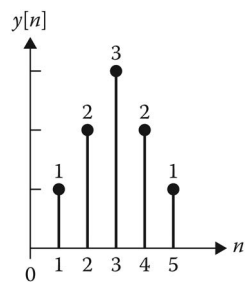


FIGURE 2.18 For Example 2.7; $y[n] = x[n]*h[n]$.

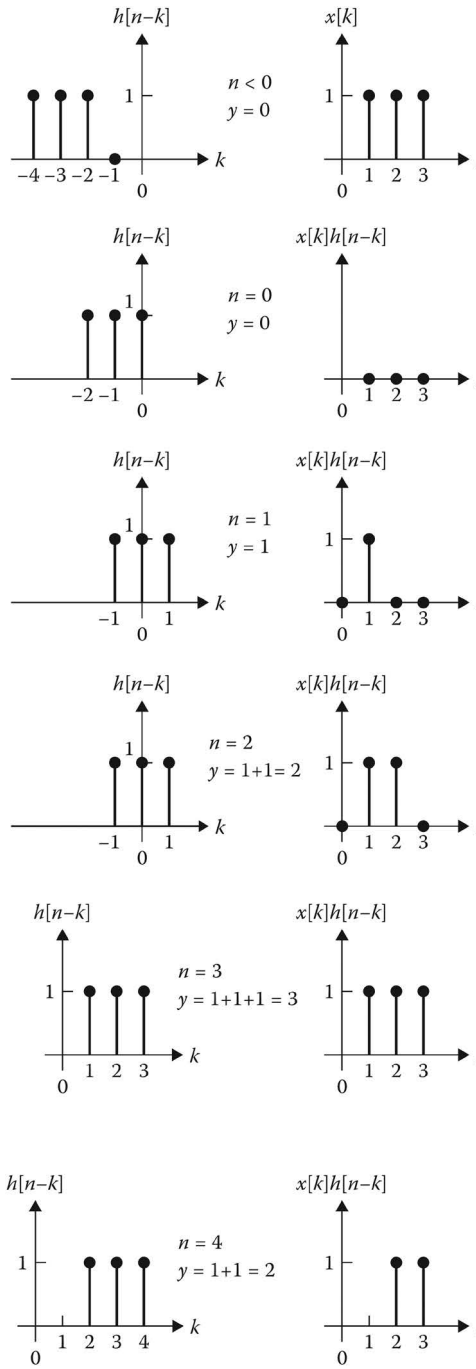


FIGURE 2.19 For Example 2.7(b); graphical convolution.

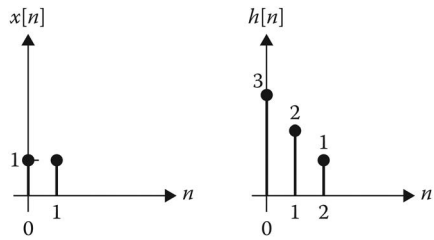


FIGURE 2.20 For Practice Problem 2.7.

Practice Problem 2.7 Find the convolution of the discrete signals shown in Figure 2.20. The signals are all zero outside the range indicated.

Answer: See Figure 2.21.

Example 2.8

Consider a discrete-time LTI system for which the input and impulse functions are given by

$$x[n] = (0.7)^n u[n], \quad h[n] = (0.2)^n u[n]$$

Find the response $y[n]$.

SOLUTION

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

But $x[k] = (0.7)^k u[k]$ and $h[n-k] = (0.2)^{n-k} u[n-k]$.

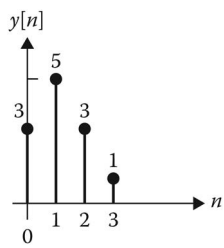


FIGURE 2.21 Solution for Practice Problem 2.7.

Because $x[n]$ and $h[n]$ are causal, we can use eq. (2.22).

$$y[n] = x[n] * h[n] = \sum_{k=0}^{n} (0.7)^k (0.2)^{n-k}, \quad n = 0,1,2,\ldots$$

$$= (0.2)^n \sum_{k=0}^{n} \left(\frac{0.7}{0.2}\right)^k$$

This is a geometric series. We notice from Appendix A,

$$\sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = \frac{1-(a/b)^{n-1}}{1-a/b}$$

Taking $a = 0.7$ and $b = 0.2$

$$y[n] = 0.2^n \frac{1-(0.7/0.2)^{n+1}}{1-0.7/0.2} = -2\left[(0.2)^{n+1} - (0.7)^{n+1}\right]$$

Therefore,

$$y[n] = 2\left[(0.7)^{n+1} - (0.2)^{n+1}\right]u[n]$$

Practice Problem 2.8 Suppose for a discrete-time LTI system, the input $x[n]$ and the impulse response $h[n]$ are

$$x[n] = 4^n u[n] \text{ and } h[n] = u[n].$$

Find the output signal $y[n]$.

Answer: $y[n] = \begin{cases} 4^{n+1}, & n < 0 \\ \dfrac{4}{3}, & n \geq 0 \end{cases}$

2.7 BLOCK DIAGRAM REALIZATION

Just as with the continuous-time case, we can find the impulse response of systems connected in series (cascade) or in parallel. Figure 2.22(a) illustrates the commutative property; i.e., $x[n]$ and $h[n]$ are interchangeable. The associative property, illustrated in Figure 2.22(b), shows that when two systems (with impulse responses $h_1[n]$ and $h_2[n]$) are connected in series, the overall impulse response is $h_1[n]*h_2[n]$. The distributive property, illustrated in Figure 2.22(c), shows that when two systems are connected in parallel, the impulse response of the composite system is $h_1[n] + h_2[n]$.

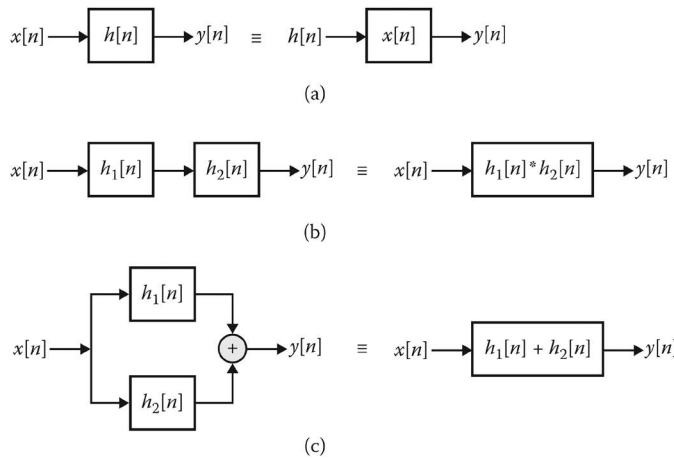


FIGURE 2.22 Block diagrams illustrating interconnected systems.

2.8 DECONVOLUTION

We know that if the impulse response $h[n]$ of a system is known, we can find the response $y[n]$ to an input $x[n]$ as simply the convolution of $x[n]$ and $h[n]$. If we know $x[n]$ and $y[n]$, how do we get $h[n]$? This process is known as *deconvolution* and also as inverse filtering or system identification.

Deconvolution is the process of obtaining one of the constituent signals in the convolution sum.

Deconvolution is often encountered in practice when trying to measure the response of the communication channel. It has no direct mathematical definition in the continuous-time domain. Discrete convolution can be done in two ways: polynomial division and recursive algorithm. Both methods will be shown in the following example. We will also use a MATLAB® command for performing discrete deconvolution.

Example 2.9

Let $x[n] = \{3, 0, 2, 6\}$ and $y[n] = \{6, 10, 25, 20, 38, 42\}$. Find $h[n]$.

SOLUTION

We will obtain the corresponding $h[n]$ in two ways.

Method 1 (Long division):
We regard the given sequences $x[n]$ and $y[n]$ as coefficients of the following polynomials in descending order.

$$x(z) = 3z^3 + 0z^2 + 2z + 6, \quad y(z) = 6z^5 + 12z^4 + 25z^3 + 20z^2 + 38z + 42$$

We now perform a long division $y(z)/x(z)$.

$$
\begin{array}{r}
2z^2+4z+7 \\
3z^3+0z^2+2z+6\,\big|\,\overline{6z^5+12z^4+25z^3+20z^2+38z+42} \\
\underline{6z^5+0z^4+4z+12z^2} \\
12z^4+21z^3+8z^2+38z+42 \\
\underline{12z^4+0z^3+8z^2+24z} \\
21z^3+0z^2+14z+42 \\
\underline{21z^3+0z^2+14v+42} \\
0
\end{array}
$$

From this, we obtain the polynomial $h(z) = 2z^2 + 4z + 7$ or $h[n] = \{2\ 4\ 7\}$.

Method 2 (recursive algorithm):
By definition,

$$y[n] = x[n] * h[n] = \sum_{k=0}^{n} h[k]x[n-k] \tag{2.9.1}$$

This convolution sum can be cast as a recursive algorithm. For $n = 0$,

$$y[0] = x[0]h[0] \quad \rightarrow \quad h[0] = y[0]/x[0] = 6/3 = 2 \tag{2.9.2}$$

We separate the term containing $x[0]$ in eq. (2.9.1).

$$y[n] = h[n]x[0] + \sum_{k=0}^{n-1} h[k]x[n-k] \tag{2.9.3}$$

From this, we obtain for $n > 0$,

$$h[n] = \frac{1}{x[0]}\left[y[n] - \sum_{k=0}^{n-1} h[k]x[n-k] \right] \tag{2.9.4}$$

We will need to evaluate this at $N_y - N_x + 1$ points, where N_y and N_x are the lengths of $y[n]$ and $x[n]$, respectively. In this example,

$$N_y = 6,\ N_x = 4 \text{ so that } N_y - N_x + 1 = 6 - 4 + 1 = 3$$

that is, we need three evaluations of $h[n]$. From eq. (2.9.2), we already got $h[0] = 2$. We now apply eq. (2.9.4) twice. For $n = 1$,

$$h[1] = \frac{1}{x[0]}\left[y[1] - \sum_{k=0}^{0} h[k]x[1-k] \right] = \frac{1}{x[0]}\left[y[1] - h[0]x[1] \right] = \frac{1}{3}[12 - 2 \times 0] = 4$$

For $n = 2$,

$$h[2] = \frac{1}{x[0]}\left[y[2] - \sum_{k=0}^{1} h[k]x[2-k] \right] = \frac{1}{x[0]}\left[y[2] - h[0]x[2] - h[1]x[1] \right]$$
$$= \frac{1}{3}[25 - 2\times 2 - 4\times 0] = 7$$

Thus, $h = \{2, 4, 7\}$ as we obtained previously.

Practice Problem 2.9 Given that $x[n] = \{1, 8, 4, 5\}$ and $y[n] = \{4, 38, 66, 61, 46, 14, 5\}$, find $h[n]$.

Answer: $h[n] = \{4, 6, 2, 1\}$.

2.9 COMPUTING WITH MATLAB®

MATLAB provides a simple way of performing convolution. Given two signals x and h, we can use MATLAB to do the convolution with the built-in function **conv**(x,h). The function **conv** returns a vector of length $L_x + L_h - 1$, where L_x is the length of x and L_h is the length of h. This produces an output that will be longer than the input. Often the additional data can be discarded.

It does not matter whether x and h are continuous-time or discrete-time signals. We use **conv** function for both cases. In fact, the function **conv** is actually designed for discrete-time convolution. To use it for continuous-time convolution, we need to evaluate the convolution integral numerically. Consider the convolution integral

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau \tag{2.23}$$

Let T, the step size or sampling period, be small and let $t = kT$ and $\tau = nT$, the convolution integral becomes a convolution summation, which can be expressed as

$$y(kT) \approx T\sum_{n=0}^{kT} x(nT)h((k-n)T) = T\sum_{n=0}^{k} x[n]h[k-n] \tag{2.24}$$

This approximates a rectangular rule integration. Equation (2.24) can be written as

$$y(k) \approx T\sum_{n=0}^{k} x[n]h[k-n] \tag{2.25}$$

which looks like discrete-time convolution except for multiplying by T.

For the purpose of illustration, we will solve one example for discrete-time signals and one example for continuous-time signals.

The MATLAB command **deconv** (*y*,*h*) can be used to find the discrete deconvolution of *y*[*n*] and *x*[*n*], producing *h*[*n*]. While **conv** can be used to multiply polynomials, **deconv** can be used to divide polynomials.

Example 2.10

Use MATLAB to find the convolution of the sequences

```
x1[n]={0.2,1.4,2.6,5.1,3.4,8.4}
x2[n]={1.0,4.2,3.7,0.8,3.9}
```

SOLUTION

The MATLAB to perform the convolution is as follows.

```
x1 = [0.2, 1.4, 2.6, 5.1, 3.4, 8.4]
x2 = [1.0, 4.2, 3.7, 0.8, 3.9]
y = conv(x1,x2)
```

The result is:

y = 0.2000 2.2400 9.2200 21.3600 36.3400 49.0900 62.0800 53.6900 19.9800 32.7600

Thus,

$x_1[n]*x_2[n]$ = {0.2000 2.2400 9.2200 21.3600 36.3400 49.0900 62.0800 53.6900 19.9800 32.7600}

Practice Problem 2.10 Using MATLAB, find the convolution of these sequences:

$x_1[n]$ = {2, 5, 4, 3, 5, 1, 0}, $x_2[n]$ = { 1, 6, 3, 9, 4, 7, 2}

Answer: *y*[*n*] = [2 17 40 60 88 110 103 98 58 45 17 2 0].

Example 2.11

A system is represented by its impulse response

$$h(t) = \frac{1}{4}(e^{-2t} - e^{-t})$$

Find and plot the response when the input is $x(t) = \cos(t)\, u(t)$.

SOLUTION

To use the function **conv** for continuous-time convolution, we need to multiply conv with a sampling period or step size, as shown in eq. (2.25). The MATLAB

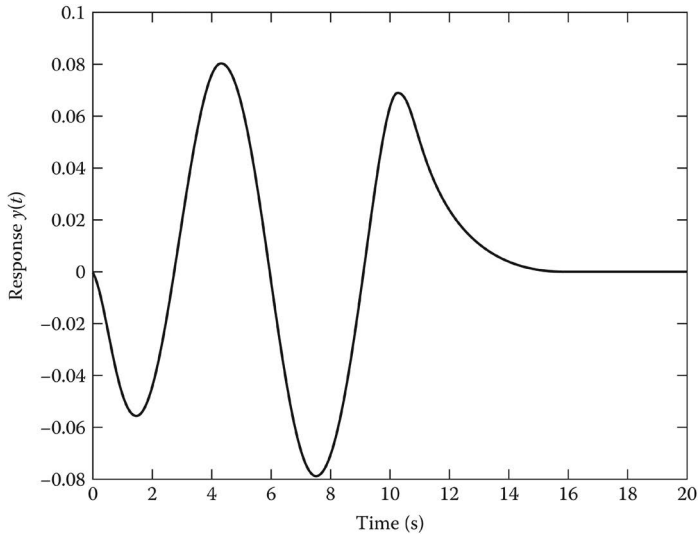


FIGURE 2.23 For Example 2.11.

code is as follows, while the plot is shown in Figure 2.23.

```
T = 0.1;  % sampling period
t = 0:T:10;
x = cos(t); % calculates x(t)
h = 0.25*( exp(-2*t) - exp(-t) );  % calculates h(t)
y = T*conv(x,h); %this contains L_x + L_h -1
t0 = (0:200)*T
plot(t0,y)      % or use this  plot(t,y(1:101))
xlabel('Time (s)')
ylabel('Response y(t)')
```

Practice Problem 2.11 Repeat Example 2.11 when

$$h(t) = \frac{1}{2}(e^{-t} - e^{-4t})u(t), \quad x(t) = e^{-t}\sin(t)u(t).$$

Answer: See Figure 2.24.

Example 2.12

Repeat Example 2.9 using MATLAB.

SOLUTION

In MATLAB, we enter the following lines:

```
x = [ 3 0 2 6 ]
y = [ 6 10 25 20 38 42 ]
h = deconv(y,x)
```

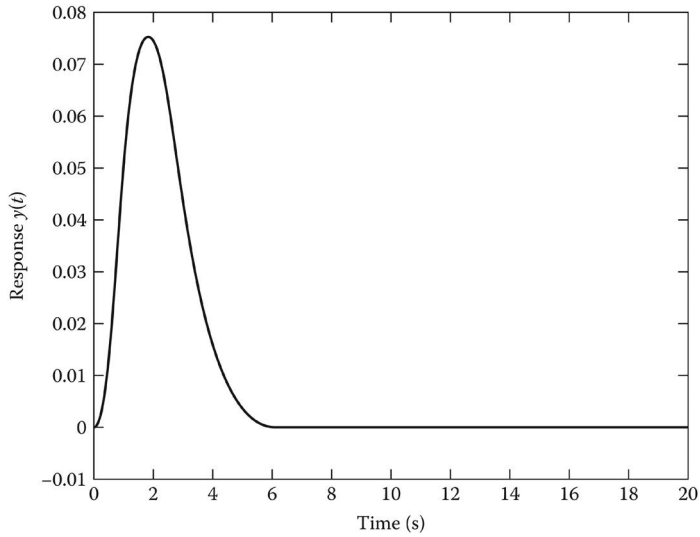


FIGURE 2.24 For Practice Problem 2.11.

This produces

h = [2 4 7]

Practice Problem 2.12 Repeat Practice Problem 2.9 using MATLAB.

Answer: h = [4 6 2 1].

2.10 APPLICATIONS

There are several applications of the convolution technique developed in this chapter. Convolution is used in circuit analysis, statistics, probability theory, optics, acoustics, digital image processing, and computational fluid dynamics. We will employ convolution to determine the stability of systems. We will also consider an application in circuit analysis.

2.10.1 BIBO Stability of Continuous-Time Systems

A system is bounded-input bounded-output (BIBO) stable if every bounded input signal produces a bounded output. We can determine the condition for stability by applying the convolution integral. Suppose we are given an LTI system and we apply a bounded input $x(t)$. The boundedness of $x(t)$ can be expressed as $|x(t)| < K$, where K is a constant. The output $y(t)$ can be bounded as follows. By definition,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

Therefore,

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right| \le \int_{-\infty}^{\infty} |x(t-\tau)||h(\tau)|\,d\tau \le \int_{-\infty}^{\infty} K|h(\tau)|\,d\tau$$
$$= K\int_{-\infty}^{\infty} |h(\tau)|\,d\tau \tag{2.26}$$

This shows that $|y(t)|$ is finite if $\int_{-\infty}^{\infty} K|h(\tau)|\,d\tau$ is finite. This reduces to $\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty$ since K is finite. For an LTI system, the condition for stability reduces to

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty \tag{2.27}$$

A continuous-time system is **BIBO stable** if its impulse response $h(t)$ is absolutely integrable.

Example 2.13

An LTI system has the impulse function $h(t) = e^{-2t}u(t)$. Determine whether the system is stable or not.

SOLUTION

Using eq. (2.27),

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau = \int_{0}^{\infty} e^{-2\tau}d\tau = \frac{e^{-2\tau}}{-2}\Bigg|_{0}^{\infty} = \frac{1}{2} < \infty$$

Since $h(t)$ is integrable, we conclude that the system is stable.

Practice Problem 2.13 Repeat Example 2.13 with $h(t) = u(t)$. This is the impulse response for a continuous-time integrator.

Answer: Unstable.

2.10.2 BIBO Stability of Discrete-Time Systems

The same analysis applies to discrete-time systems. Suppose we have an LTI system and we apply a bounded input $x(t)$, with $|x[n]| < K$, where K is a constant. We can bound the output as follows. We recall that

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Therefore,

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[n-k]h[k]\right| \le \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \le \sum_{k=-\infty}^{\infty} K|h[k]|$$
$$= K\sum_{k=-\infty}^{\infty} |h[k]| \tag{2.28}$$

This shows that $|y[n]|$ is finite if $\sum_{k=-\infty}^{\infty} |h[k]|$ is finite. Thus, for stability ($|y[n]| < \infty$)

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \tag{2.29}$$

A discrete-time system is **BIBO stable** if its impulse response $h[n]$ is absolutely summable.

Example 2.14

A discrete-time system has an impulse response $h[n] = \frac{1}{n}u[n]$. Determine if the system is stable.

SOLUTION

From eq. (2.29)

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} \frac{1}{n} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{n}$$

which does not converge as $N \to \infty$. Thus, $h[n]$ is not absolutely summable, and we conclude that the system is unstable.

Practice Problem 2.14 A filter is described by $h[n] = (-0.6)^n$. Is the system stable?

Answer: Stable.

2.10.3 Circuit Analysis

We now apply convolution to analyze electric circuits. The following example will illustrate the procedure.

Example 2.15

The pulse $v_s(t)$ in Figure 2.25(a) is applied to the RL circuit in Figure 2.25(b). Use the convolution integral to find the response $v_o(t)$.

SOLUTION

This problem requires us to know the impulse response of the circuit. Throughout this chapter, we assume that the impulse response $h(t)$ is given. We will learn how to find it in the next chapter. For now, let us assume that the impulse response of the circuit is

$$h(t) = e^{-t}\,u(t)$$

From Figure 2.25(a), $v_s(t) = u(t) - u(t-4)$. Once we know the impulse response $h(t)$ and input signal $v_s(t)$, we can find the response $v_o(t)$.

$$\begin{aligned} v_o(t) = h(t) * v_s(t) &= \int_0^t v_s(\tau)h(t-\tau)d\tau \\ &= \int_0^t \big[u(\tau) - u[\tau-4]\big]e^{-(t-\tau)}d\tau \end{aligned} \tag{2.15.1}$$

Since $u(\tau - 4) = 0$ for $0 < \tau < 4$, the integrand involving $u(\tau)$ is nonzero for all τ, whereas the integrant involving $u(\tau - 4)$ is nonzero only for $\tau > 4$. The best way to handle the integral is to evaluate the two parts separately. For $0 < t < 4$,

$$\begin{aligned} v_o' &= \int_0^t (1)e^{-(t-\tau)}d\tau = e^{-t}\int_0^t (1)e^{\tau}d\tau \\ &= e^{-t}(e^t - 1) = 1 - e^{-t}, \quad 0 < t < 4 \end{aligned} \tag{2.15.2}$$

For $t > 4$,

$$\begin{aligned} v_o'' &= \int_4^t (1)e^{-(t-\tau)}d\tau = e^{-t}\int_4^t (1)e^{\tau}d\tau \\ &= e^{-t}(e^t - e^4) = 1 - e^4e^{-t}, \quad t > 4 \end{aligned} \tag{2.15.3}$$

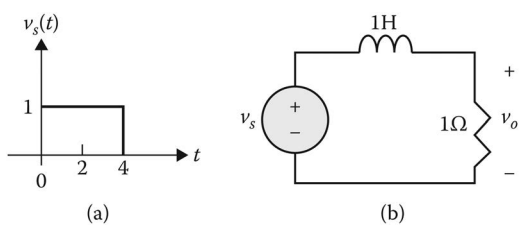


FIGURE 2.25 For Example 2.15.

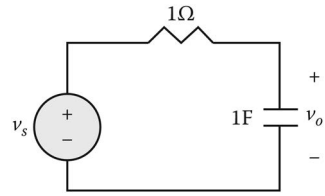


FIGURE 2.26 For Practice Problem 2.15.

Substituting eqs. (2.15.2) and (2.15.3) into eq. (2.15.1) gives

$$\begin{aligned} v_o(t) &= v_o'(t) - v_o''(t) \\ &= (1-e^{-t})\big[u(t-4)-u(t)\big] - (1-e^{4}e^{-t})u(t-4) \\ &= \begin{cases} 1-e^{-t}, & 0<t<4 \\ e^{4}e^{-t}-1, & t>4 \end{cases} \end{aligned} \tag{2.15.4}$$

Practice Problem 2.15 Find the response $v_o(t)$ of the RC circuit in Figure 2.26. Assume that the impulse response of the circuit is $h(t) = e^{-t}u(t)$.

Answer: 5 $(e^{-t} - e^{-2t})\, u(t)$ V.

SUMMARY

1. This chapter discusses time-domain analysis of LTI continuous-time and discrete-time systems.
2. Every LTI system is characterized by its impulse response $h(t)$ or $h[n]$.
3. The analysis of an LTI system can be reduced to finding the response of the system to input signals. Given the impulse response $h(t)$, the system's response $y(t)$ to an input signal $x(t)$ is given by the convolution integral:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

   Thus, convolution is a time-domain technique for determining the response of a system to an input.
4. Convolution is obtained analytically or graphically by performing four operations: (1) folding or time reversal to get $h(-\tau)$; (2) shifting to get $h(t-\tau)$; (3) multiplying $h(t-\tau)$ and $x(\tau)$; and (4) integrating $h(t-\tau)x(\tau)$.
5. The composite impulse response $h(t)$ of two systems [with impulses $h_1(t)$ and $h_2(t)$] in series is the convolution of the individual impulses, i.e., $h(t) = h_1(t)*h_2(t)$.
6. The composite impulse response $h(t)$ of two systems [with impulses $h_1(t)$ and $h_2(t)$] in parallel is the sum of the individual impulses, i.e., $h(t) = h_1(t) + h_2(t)$.

7. Block diagram representation helps us understand the dynamic behavior of systems.
8. For discrete-time systems, the impulse response $h[n]$, the input $x[n]$, and the output $y[n]$ are related by the convolution sum:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

9. Convolution (both in continuous-time and discrete-time) is commutative, associative, and distributive.
10. Deconvolution is the process of finding the impulse response $h[n]$ of a system given its input $x[n]$ and its output $y[n]$.
11. The function **conv**(x,h) is used in MATLAB® to perform both continuous-time and discrete-time convolution. The MATLAB function **deconv**(y,x) performs discrete deconvolution.
12. A continuous-time system is BIBO stable if its impulse response is absolutely integrable. A discrete-time system is BIBO stable if its impulse response is absolutely summable.
13. A simple application of convolution is circuit analysis. If we know the impulse response of a circuit, we can determine the response to any input with the use of convolution.

REVIEW QUESTIONS

2.1 It is improper to talk about an impulse response for a non-LTI system.
(a) True, (b) False

2.2 Which of these is not true?
(a) $x(t)*h(t) = h(-t)*x(t)$
(b) $x(t)*\delta(t) = x(t)$
(c) $x(t)*\delta'(t) = x'(t)$
(d) $u(t)*\delta'(t) = \delta(t)$

2.3 The methods introduced in this chapter are known as the time domain method because they involve functions of time.
(a) True, (b) False

2.4 When two systems with impulse responses $h_1(t)$ and $h_2(t)$ are connected in series, the impulse response of the composite system is:
(a) $h_1(t) + h_2(t)$, (b) $h_1(t) * h_2(t)$, (c) $h_1(t)/h_2(t)$, (d) none of the above

2.5 Consider the two signals in Figure 2.27. They are convolved to get $y(t)$. The duration of $y(t)$ is:
(a) $-1 < t < 2$, (b) $-2 < t < 2$, (c) $-1 < t < 3$, (d) none of the above

2.6 From the previous question, the area of $y(t)$ is:
(a) 2, (b) 4, (c) 6, (d) none of the above

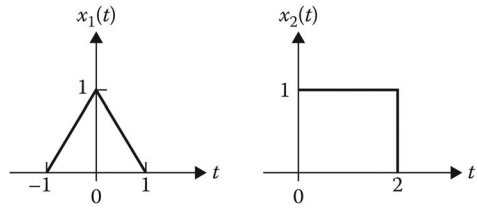


FIGURE 2.27 For Review Questions 2.5 and 2.6.

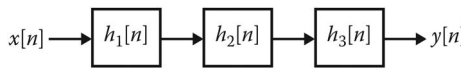


FIGURE 2.28 For Review Question 2.9.

2.7 Which of the following is not true?
(a) $\delta[n] = u[n] - u[n-1]$
(b) $u[n] = \sum_{k=-\infty}^{n} \delta[k]$
(c) $x[n]*\delta[n] = x[n]$
(d) $x[n]*h[n] = -h[n]*x[n]$

2.8 If two sequences of length 8 and 10 are convolved to give $y[n]$. The length of $y[n]$ is:
(a) 8, (b) 10, (c) 17, (d) 18, (e) none of the above

2.9 Consider the interconnection of three LTI systems in Figure 2.28. The overall impulse response is:
(a) $h_1[n] + h_2[n] + h_3[n]$
(b) $h_1[n]*h_2[n]*h_3[n]$
(c) None of the above

2.10 The MATLAB function **conv** is used for:
(a) continuous-time convolution
(b) discrete-time convolution
(c) both (a) and (b)

Answers: 2.1(a), 2.2(a), 2.3(a), 2.4(b), 2.5(c), 2.6 (a), 2.7(d), 2.8(c), 2.9(b), 2.10(c).

PROBLEMS

Section 2.3: Convolution Integral

2.1 Show that $x_1(t) * x_2(t) = x_2(t) * x_1(t)$.

2.2 Show that:
(a) $f(t)*\delta(t) = f(t)$
(b) $f(t)*\ \delta(t - t_o) = f(t - t_o)$
(c) $f(t) * u(t) = \int_{-\infty}^{t} f(\tau) d\tau$

2.3 Show $x(t) * \frac{d}{dt}\delta(t) = \frac{d}{dt}x(t)$.

2.4 Given the following signals:

$$x(t) = 2\delta(t), \quad y(t) = 4u(t), \quad z(t) = e^{-2t}u(t),$$

evaluate the following operations:
(a) $x(t)*y(t)$
(b) $x(t)*z(t)$
(c) $y(t)*z(t)$
(d) $y(t)*[y(t) + z(t)]$

2.5 Use the convolution integral to find:
(a) $t*e^{at}\,u(t)$
(b) $\cos(t)*\cos(t)u(t)$

2.6 Write the expression for $y(t) = x(t)*\delta(t-2)$ when:
(a) $x(t) = t^2$
(b) $x(t) = t(t+0.5)^2$
(c) $x(t) = 4e^{-2t}\cos(2\pi t - \pi/2)$

2.7 Sketch these functions
(a) $x(t) = \prod(t/2)*[\delta(t+1) + \delta(t+2)]$
(b) $y(t) = [\Lambda(t+1) - \Lambda(t-1)]*\delta(t)$
(c) $z(t) = \prod(t/2)*\Lambda(t)$

2.8 Show that:
(a) $u(t) * u(t) = r(t)$
(b) $u(t-1) * u(t-3) = r(t-4)$
(c) $\prod(t/\tau) * \prod(t/\tau) = \tau\Lambda(t/\tau)$

2.9 For an LTI system, the impulse response is $h(t) = e^{-2t}u(t)$. If the input to the system is $x(t) = 10e^{-2t}u(t)$, find the output $y(t)$.

2.10 The input to an LTI system is $x(t) = \cos(2t)$, while the impulse response of the system is

$$h(t) = e^{-|t|} = \begin{cases} e^{t}, & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

Determine the system's output $y(t)$.

2.11 The impulse response of a filter is $h(t) = e^{-2t}u(t) - \delta(t)$. Find the response of the filter to the input $e^{-t}u(t)$.

2.12 An LTI system has its impulse response $h(t) = u(t)$. Find the response corresponding to the input $x(t) = e^{-2t}u(t)$.

2.13 Given that the impulse response of a system is $h(t) = e^{-2t}u(t)$, determine the system's response $y(t)$ if the input is:
(a) $u(t)$
(b) $e^{-t}u(t)$
(c) $\cos(2t)u(t)$

2.14 The impulse response of a lowpass filter is $h(t) = e^{-t}u(t)$. Determine its step response, i.e., the output when the input is a unit step.

2.15 If $h(t)$ is the impulse response of a system and $x(t)$ is the input, what kinds of systems are described by: (a) $h(t) = \delta(t)$, (b) $h(t) = \dfrac{d}{dt}\delta(t)$, (c) $h(t) = u(t)$.

2.16 Given that $h(t) = 4e^{-2t}u(t)$ and $x(t) = \delta(t) - 2\,e^{-2t}u(t)$, find $y(t) = x(t)*h(t)$.

2.17 If $x(t) = u(t) - u(t-2)$, determine $y(t) = x(t)*x(t)$.

2.18 Let $x(t) = y(t) = 2e^{-3t}u(t)$. Find $x(t)*y(t)$.

2.19 Given that $x_1(t) = 4e^{-2t}u(t)$ and $x_2(t) = e^{-t}u(t)$, find the convolution of $x_1(t)$ and $x_2(t)$.

Section 2.4: Graphical Convolution

2.20 Verify the area property of convolution in Example 2.3; i.e., if A_{x1} and A_{x2} are the areas under $x_1(t)$ and $x_2(t)$, respectively, the area A_y under $y(t) = x_1(t)*x_2(t)$ is $A_y = A_{x1}A_{x2}$.

2.21 Given $x(t)$ and $h(t)$ in Figure 2.29, find $y(t) = x(t)*h(t)$.

2.22 Perform a graphical convolution of the two signals in Figure 2.30.

2.23 Determine the convolution $x_1(t)*x_2(t)$ for each pair of signals in Figure 2.31.

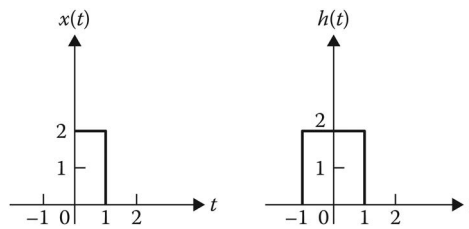


FIGURE 2.29 For Problem 2.21.

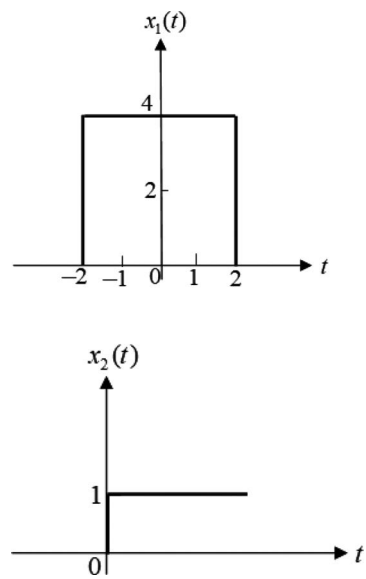


FIGURE 2.30 For Problem 2.22.

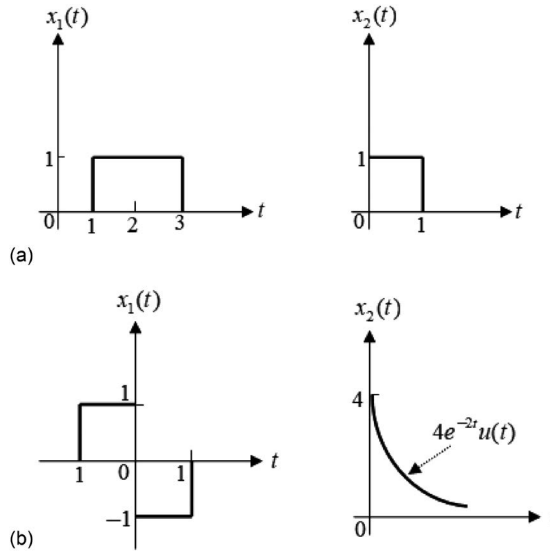


FIGURE 2.31 For Problem 2.23.

2.24 Obtain the convolution $z(t)$ of $x(t)$ and $y(t)$ shown in Figure 2.32.
2.25 Perform a graphical convolution of $x_1(t)$ and $x_2(t)$ depicted in Figure 2.33.
2.26 Find the convolution of $x(t)$ and $y(t)$ in Figure 2.34.
2.27 Given the two signals shown in Figure 2.35, perform a graphical convolution of the signals.
2.28 Given signals $x(t)$ and $y(t)$ in Figure 2.36, determine their convolution.
2.29 Consider signals $x(t)$ and $y(t)$ in Figure 2.37, find $z(t) = x(t)*y(t)$.

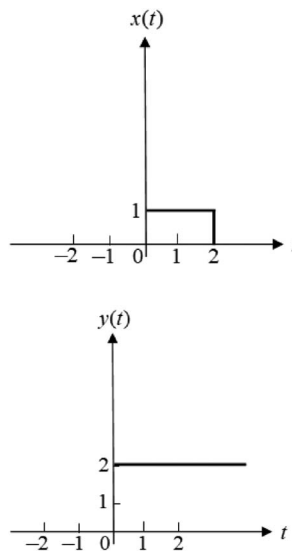


FIGURE 2.32 For Problem 2.24.

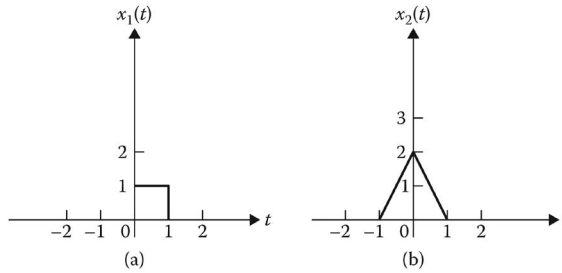


FIGURE 2.33 For Problem 2.25.

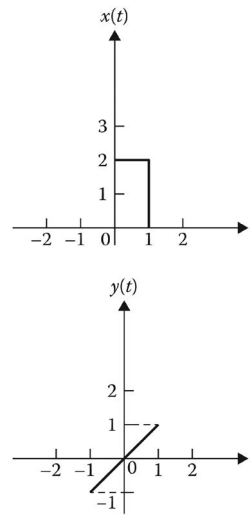


FIGURE 2.34 For Problem 2.26.

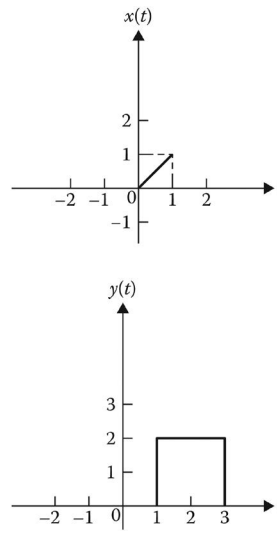


FIGURE 2.35 For Problem 2.27.

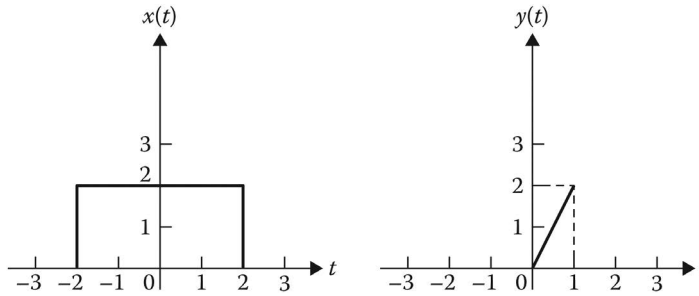


FIGURE 2.36 For Problem 2.28.

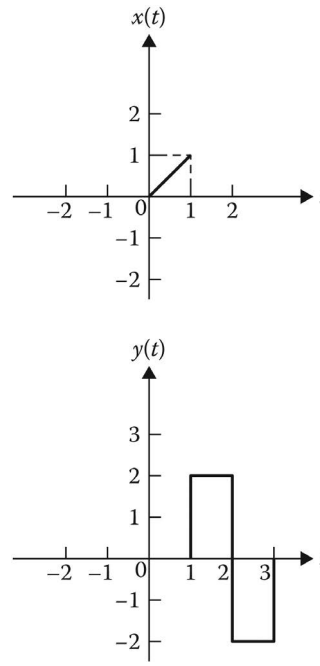


FIGURE 2.37 For Problem 2.29

Section 2.5: Block Diagram Representation

2.30 Two systems are connected in series as shown in Figure 2.38, where

$$h_1(t) = e^{-t}u(t), \quad h_2(t) = 2e^{-3t}u(t)$$

Find the impulse response of the overall systems.

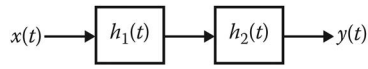


FIGURE 2.38 For Problems 2.30 and 2.31.

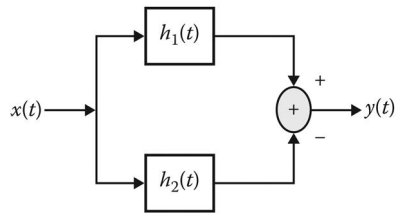


FIGURE 2.39 For Problem 2.32.

2.31 For the system shown in Figure 2.38,

$$h_1(t) = 2e^{-t}u(t), \quad h_2(t) = 4e^{-2t}u(t)$$

Determine the overall impulse response $h(t)$.

2.32 Two systems are connected in parallel as in Figure 2.39. If the impulse responses of the systems are given by

$$h_1(t) = 4e^{-t}u(t) \quad \text{and} \quad h_1(t) = e^{-2t}u(t)$$

Find the impulse response of the overall system.

2.33 Determine the overall impulse response for the system shown in Figure 2.40.

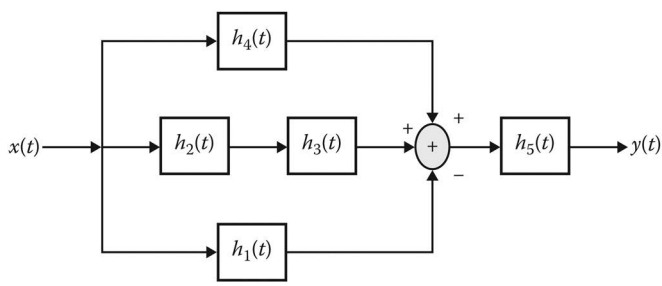


FIGURE 2.40 For Problem 2.33.

Section 2.6: Discrete-Time Convolution

2.34 The following equalities are useful in our studies. Prove each of them.

(a) $$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases}$$

(b) $$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}, \quad |a| < 1$$

(c) $$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \quad |a| < 1$$

2.35 Consider a discrete-time system with impulse response

$$h[n] = (0.4)^n u[n]$$

If $y[n]$ is the output of the system due to the input

$$x[n] = \delta[n-1] + 3\delta[n]$$

find $y[2]$ and $y[5]$.

2.36 Determine $y[n] = x[n]*h[n]$ for the following pairs of signals:

(a) $x[n] = u[n]$, $h[n] = 4^n u[n]$

(b) $x[n] = h[n] = 2^n u[n]$

(c) $x[n] = (0.3)^n u[n]$, $h[n] = 2^n u[n]$

2.37 Find the convolution of the signals $x_1[n]$ and $x_2[n]$ given as:

$$x_1[n] = 2(3)^n u[n], \quad x_2[n] = 3(2)^n u[n].$$

2.38 Evaluate the convolution of $x[n]$ and $h[n]$ shown in Figure 2.41.

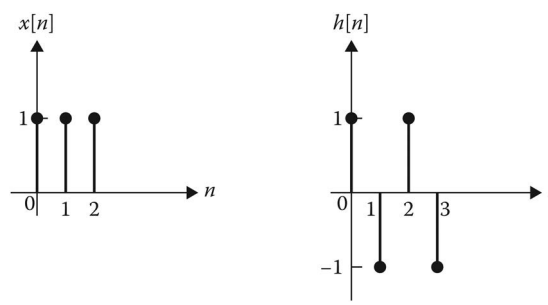


FIGURE 2.41 For Problem 2.38.

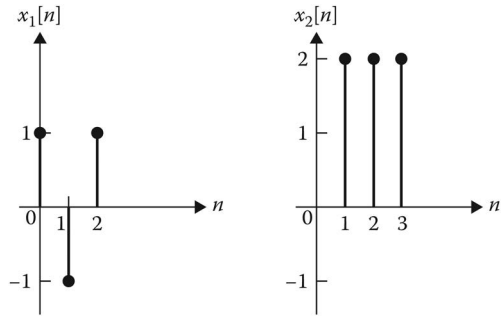


FIGURE 2.42 For Problem 2.40.

2.39 Given that $x[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$, $h[n] = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 2, & n = 3 \\ 0, & \text{otherwise} \end{cases}$

(a) Sketch $x[n]$ and $h[n]$.
(b) Find $x[n]*h[n]$.

2.40 Determine the convolution of the discrete signals in Figure 2.42. Assume that the signals are zero outside the range indicated.

2.41 Obtain the convolution of the discrete signals in Figure 2.43. Sketch your result. Assume that the signals are zero outside the range indicated.

2.42 (a) Find the convolution of the discrete signals in Figure 2.44(a)
(b) Repeat part (a) for the signals in Figure 2.44(b).

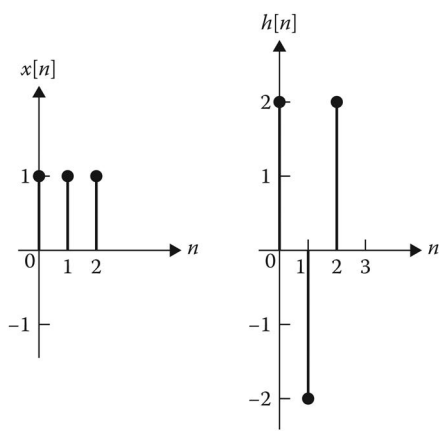


FIGURE 2.43 For Problem 2.41.

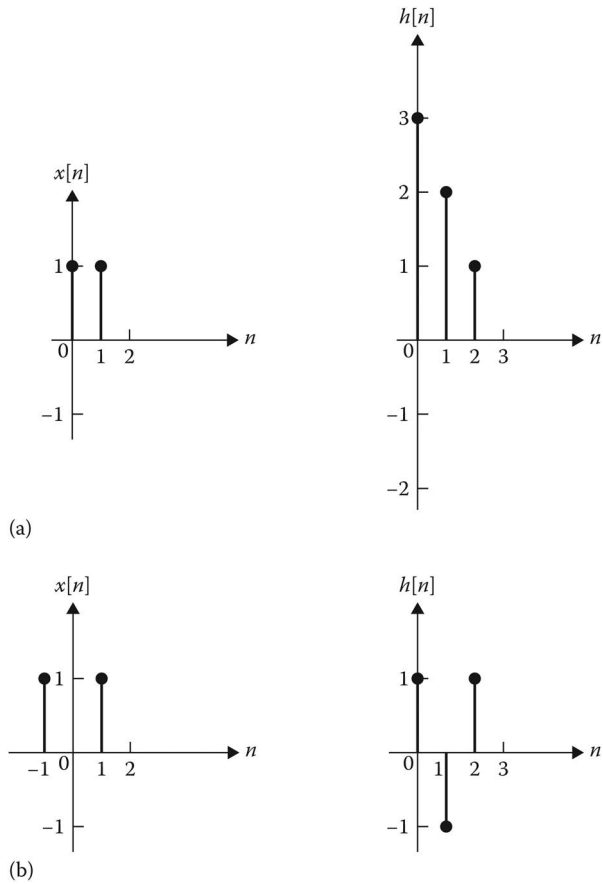


FIGURE 2.44 For Problem 2.42.

2.43 Determine the convolution of the discrete signals in Figure 2.45.

2.44 For the signals $x[n]$ and $y[n]$ in Figure 2.46, find $z[n] = x[n]*y[n]$.

2.45 Given that $x[n] = u[n]$ and $h[n] = (0.4)^n + (0.5)^{n+1}$, $n \geq 0$. Find $y[n] = x[n]*h[n]$.

$$\text{Hint: } \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

2.46

Let $x[n] = \delta[n+2] + 2\delta[n+1] + \delta[n] + \delta[n-1]$,

$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$,

find $y[n] = x[n] * h[n]$.

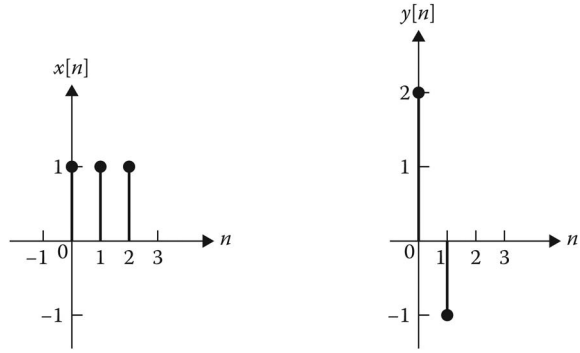


FIGURE 2.45 For Problem 2.43.

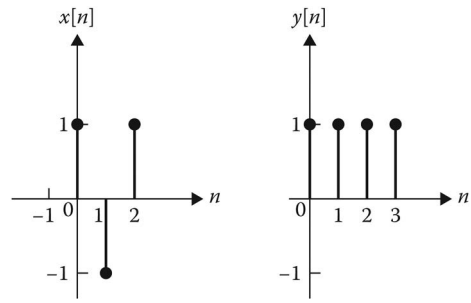


FIGURE 2.46 For Problem 2.44.

Section 2.7: Block Diagram Realization

2.47 Two systems are described by

$$h_1[n] = (0.4)^n u[n], \quad h_2[n] = \delta[n] + 0.5\delta[n-1]$$

Determine the response to the input $x[n] = (0.4)^n u[n]$ if:
(a) The two systems are connected in parallel
(b) The two systems are connected in cascade

2.48 For the interconnection of LTI systems in Figure 2.47, find the overall impulse response.

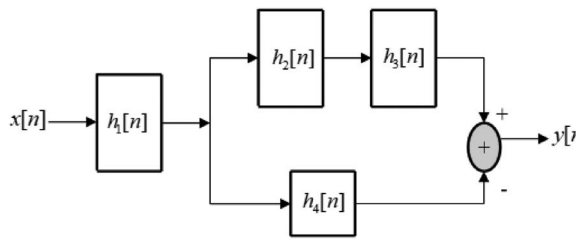


FIGURE 2.47 For Problem 2.48.

Section 2.8: Deconvolution

2.49 Find the impulse response of a system which yields $y[n] = [2\ 3\ -2\ 4]$ when excited by the input $x[n] = [1\ 0\ 2]$.

2.50 The input $x[n] = [1\ -1]$ to a system produces the output $y[n] = [4\ 2\ 5\ 1]$. Determine the impulse response.

Section 2.9: Computing with MATLAB

2.51 (a) Using MATLAB, write a program to convolve $x(t) = e^{-t}u(t)$ and $h(t) = u(t) - u(t-2)$.
(b) Do the convolution analytically and compare this with the MATLAB result in part (a).

2.52 A filter has the impulse response as

$$h[n] = \{1,1,1,1,1,-1,-1,-1,-1,-1\}$$

Use MATLAB to determine the output due to $x[n] = \cos(\pi n/2)$.

2.53 An LTI discrete system has the impulse response $h[n] = (0.6)^n u[n]$. Use MATLAB to calculate the response of the system to input $x[n] = u[n]$ and plot it.

2.54 Repeat the previous problem for $x[n] = \cos(n\pi/6)u[n]$.

2.55 Given that $x[n] = [1\ -1\ 2\ 4]$ and $y[n] = [2\ 6\ 4\ 0\ 8\ 5\ 12]$, use MATLAB to find $h[n]$.

Section 2.10: Applications

2.56 Determine the stability of the LTI system with the following impulse responses.

(a) $h(t) = e^{2t}u(t)$

(b) $h(t) = \sin 2t\ u(t)$

(c) $h(t) = e^{-t}\cos 2t\ u(t)$

2.57 A system has an impulse response $h(t) = u(t+1) - u(t-1)$. Is the system stable?

2.58 State which of the systems represented by the following impulse responses are stable or unstable.

(a) $h[n] = \delta[n]$

(b) $h[n] = (-0.5)^n u[n]$

(c) $h[n] = u[n] - u[n-10]$

2.59 An accumulator has impulse response $h[n] = u[n]$. Check if the accumulator is BIBO stable.

2.60 The input to the circuit discussed in Example 2.15 is $v_s(t) = u(t-1) + \delta(t-2)$. Use convolution to determine the output.

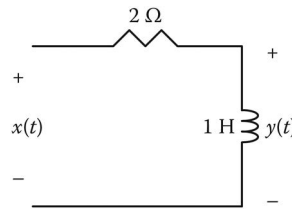


FIGURE 2.48 For Problem 2.62.

2.61 Determine the output of the circuit discussed in Practice Problem 2.15 if $v_s(t) = u(t)$.

2.62 If the filter in Figure 2.48 has the impulse response $h(t) = \delta(t) - 2e^{-2t}u(t)$, find the step response.

3 The Laplace Transform

What we know is not much. What we do not know is immense.

Pierre Simon Laplace

HISTORICAL PROFILE

Pierre Simon Laplace (1749–1827) was a French astronomer and mathematician whose work was important to the development of mathematical astronomy and statistics. He also put the theory of mathematical probability on a sound footing. He first discovered Laplace's equation and Laplace transform, to be discussed in this chapter. To Laplace, the universe is nothing but a giant problem in calculus.

Laplace was born at Beaumont-en-Auge, Normandy, France, on March 23, 1749, and died at Paris on March 5, 1827. Laplace became a professor of mathematics at the age of 20. He is remembered as one of the greatest scientists of all time. He

DOI: 10.1201/9781003490869-3

is sometimes referred to as the Newton of France. He was widely known for his five-volume work, *Celestial Mechanics*, which supplemented the work of Newton on astronomy. After the publication of this work, Laplace continued to apply his ideas of physics to other problems such as capillary action, double refraction, the velocity of sound, the theory of heat, rotation of the cooling earth, and elastic fluids. He was born and died a Catholic.

3.1 INTRODUCTION

We saw in Chapter 2 that system analysis in the time-domain involves the evaluation of the convolution integral or sum since the response of the system is given by the convolution of the input and the impulse response. This chapter introduces the *Laplace transform,* a very powerful, alternative tool for analyzing systems. The Laplace transform is named after Pierre Simon Laplace (1749–1827), the French astronomer and mathematician. In contrast to the time-domain models studied in the previous chapter, the Laplace transform is a frequency-domain representation that makes analysis and design of linear systems simpler. It is just one kind of the many transforms out there in the world.

The Laplace transform is a well-established tool in analyzing continuous-time linear systems. It is an integral transform that converts a function $x(t)$ in the time domain to a function $X(s)$ in the frequency domain, also known as the ***s*-domain**. It is important in science and engineering for a number of reasons. First, it is applicable to a wider variety of inputs. The Laplace transform of an unbounded signal can be found. Second, the Laplace transform is powerful in providing us in one single operation the complete response, that is, the steady-state plus transient or homogenous and particular. Third, it is a powerful tool for solving differential equations. It automatically includes the initial conditions in the system analysis. This allows us to convert ordinary differential equations into algebraic equations which are easier to manipulate and solve. It converts convolution into a simple multiplication. Fourth, we can apply the Laplace transform to generate the transfer function representation of a continuous-time LTI system.

The chapter begins with the definition of the Laplace transform and uses that definition to derive the transform of some basic important functions. We will consider some properties of the Laplace transform which are helpful in obtaining the Laplace transform of other functions. We then consider the inverse Laplace transform. We apply all these to solving integrodifferential equations and system analysis, especially in circuit and control systems. We finally demonstrate how MATLAB can be used to do most of what we cover in this chapter.

3.2 DEFINITION OF THE LAPLACE TRANSFORM

There are two types of Laplace transforms.

1. The two-sided (or bilateral) Laplace transform
   The two-sided Laplace transform allows time functions to be non-zero for

negative time.

$$\mathcal{L}\left[x(t)\right] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{3.1}$$

where s is the complex frequency given by

$$s = \sigma + j\omega \tag{3.2}$$

where σ (in nepers/second) and ω (in radians/second) are the real and imaginary parts of s, respectively. The advantage of the bilaterial Laplace transform is that it can handle both causal and non-causal signals over $-\infty$ to ∞.

2. The one-sided (or unilateral) form
   A one-sided Laplace transform is zero for negative time; that is, $t < 0$. It is found from eq. (3.1) by setting the lower limit of the integral equal to zero.

$$\mathcal{L}\left[x(t)\right] = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \tag{3.3}$$

where 0^- denotes a time just before 0.

The one-sided Laplace transform is more commonly used than the two-sided Laplace transform because we encounter only positive-time signals in practice and it is defined for positive-time signals.

According to eq. (3.3), the t-domain signal $x(t)$ is changed into the s-domain function $X(s)$.

The **Laplace transform** of a signal $x(t)$ is the integration of the product of $x(t)$ and e^{-st} over the interval from 0 to $+\infty$.

If $x_1(t)$ and $x_2(t)$ have the same transform then

$$\mathcal{L}[x_1(t)] = \mathcal{L}[x_2(t)] = X(s) \tag{3.4}$$

A signal $x(t)$ is Laplace transformable, or the Laplace transform exists if $x(t)$ satisfies the following inequality:

$$\int_{0^-}^{\infty} |x(t)| e^{-\sigma t}dt < \infty \tag{3.5}$$

for some real value of $\sigma = \sigma c$. Hence, the region of convergence (ROC) for the Laplace transform is $Re(s) = \sigma > \sigma c$, as shown in Figure 3.1, and the ROC means the range

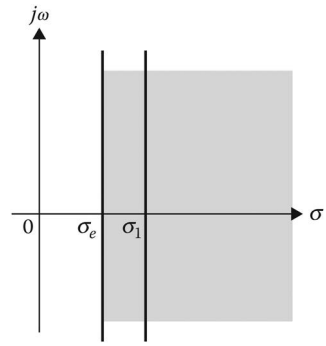


FIGURE 3.1 Region of convergence for the Laplace transform.

of s for which the Laplace transform converges. We cannot define $X(s)$ outside the ROC. The portion of the s plane in which the integral converges is found by the value of σ. All signals that we come across within the system analysis fulfills the convergence criterion in eq. (3.5) and have their corresponding Laplace transforms.

In the inverse Laplace transform the s-domain $X(s)$ is changed back to the t-domain $x(t)$ and is given as

$$\mathcal{L}^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} X(s)e^{st}ds \tag{3.6}$$

where the integration operates along a straight line $(\sigma_1 + j\omega,\ -\infty < \omega < \infty)$, as shown in Figure 3.1. The Laplace transform pair $x(t)$ and $X(s)$ are represented as

$$x(t) \Leftrightarrow X(s) \tag{3.7}$$

because there is one-to-one correspondence between $x(t)$ and $X(s)$.

Example 3.1

Find the Laplace transform of the following functions and establish the region of convergence (ROC) for each case.

a. $e^{-5t}u(t)$
b. $\delta(t)$

SOLUTION

a. For the exponential function, shown in Figure 3.2(a), the Laplace transform is

$$\mathcal{L}[e^{-5t}u(t)] = \int_{0^-}^{\infty} e^{-5t}e^{-st}dt = -\frac{1}{s+5}e^{-(s+5)t}\Big|_{0^-}^{\infty} = \frac{1}{s+5} \tag{3.1.1}$$

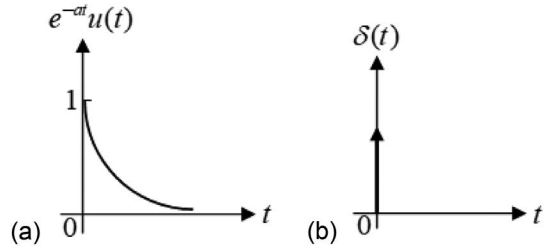


FIGURE 3.2 For Example 3.1: (a) exponential function, (b) unit impulse function.

The result in eq. (3.1.1) is so important that it deserves boxing. In general

$$\mathcal{L}[e^{-at}u(t)] = \frac{1}{s+a} \tag{3.1.2}$$

A special case of this is when $a = 0$. $e^{-at}u(t)$ becomes $u(t)$ and we have

$$\mathcal{L}[u(t)] = \frac{1}{s} \tag{3.1.3}$$

b. For the unit impulse function, shown in Figure 3.2(b),

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-0} = 1 \text{ for all } s \tag{3.1.4}$$

The sifting property in eq. (1.28) has been applied in eq. (3.1.4). In this case, the transform does not depend on s and hence the region of convergence is the entire s plane.

Practice Problem 3.1 Determine the Laplace transform of these functions and specify the ROC for each:

a. $h(t) = \frac{t^2}{2}u(t)$,

b. $e^{-j\omega t}u(t)$.

Answer: (a) $1/s^3$, $Re(s) > 0$, (b) $1/(s + j\omega)$, $Re(s) > 0$.

Example 3.2

Find the Laplace transform of $x(t) = \sin \omega t\, u(t)$.

SOLUTION

The Laplace transform of the cosine function is

$$X(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t) e^{-st} dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right) dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

Practice Problem 3.2 Find the Laplace transform of $x(t) = 10 \cos \omega t$.

Answer: $\dfrac{10\omega}{s^2 + \omega^2}$.

3.3 PROPERTIES OF THE LAPLACE TRANSFORM

The properties of the Laplace transform are summarized next, which are useful to obtain the transform of many signals. Some of the properties are linearity, scaling, time shifting, frequency shifting, time differentiation, time convolution, time integration, frequency differentiation, time periodicity, modulation, and initial and final values.

3.3.1 Linearity

Let the Laplace transform of two signals $x_1(t)$ and $x_2(t)$ be $X_1(s)$ and $X_2(s)$, respectively. Then the linearity property states that the Laplace transform of the sum of two functions of time is equal to the sum of the transforms of each function and is given as

$$\mathcal{L}\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 X_1(s) + a_2 X_2(s) \tag{3.8}$$

where a_1 and a_2 are constants.

The linearity property is explained as follows:

$$\begin{aligned} \mathcal{L}\left[a_1 x_1(t) + a_2 x_2(t)\right] &= \int_0^\infty \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-at}\, dt \\ &= a_1 \int_0^\infty x_1(t) e^{-at}\, dt + a_2 \int_0^\infty x_2(t) e^{-at}\, dt \\ &= a_1 X_1(s) + a_2 X_2(s) \end{aligned} \tag{3.9}$$

For example, consider

$$x(t) = 3e^{4t} + 5e^{-6t}. \tag{3.10}$$

By using the linearity property,

$$X(s) = \mathcal{L}[3e^{4t} + 5e^{-6t}] = \mathcal{L}\left[3e^{4t}\right] + \mathcal{L}\left[5e^{-6t}\right] \tag{3.11}$$

we obtain

$$X(s) = \frac{3}{s-4} + \frac{5}{s+6} \tag{3.12}$$

3.3.2 Scaling

Let $X(s)$ be the Laplace transform of the signal $x(t)$. Let a be any positive real constant. Then the Laplace transform of $x(at)$ is given as

$$\mathcal{L}\left[x(at)\right] = \int_0^\infty x(at)\, e^{-st}\, dt \tag{3.13}$$

Let $\lambda = at$, $d\lambda = a\, dt$, then we obtain

$$\mathcal{L}\left[x(at)\right] = \int_0^\infty x(\lambda)\, e^{-\lambda\left(\frac{s}{a}\right)} \frac{d\lambda}{a} = \frac{1}{a}\int_0^\infty x(\lambda)\, e^{-\lambda\left(\frac{s}{a}\right)}\, d\lambda \tag{3.14}$$

Thus, the scaling property is obtained as

$$\mathcal{L}\left[x(at)\right] = \frac{1}{a} X\left(\frac{s}{a}\right), \quad a > 0 \tag{3.15}$$

The unilateral Laplace transform is valid only for causal signals and is not valid for non-causal signal (when a is negative, i.e., $a < 0$).

For example, consider

$$\mathcal{L}[r(t)] = \frac{1}{s^2} \tag{3.16}$$

By applying the scaling property,

$$\mathcal{L}[r(2t)] = \frac{1}{2}\frac{1}{(s/2)^2} = \frac{2}{s^2} \tag{3.17}$$

3.3.3 Time Shifting

Often we can see that there is a signal delay in control systems due to transmission delays or other effects. Let $X(s)$ be the Laplace transform of signal $x(t)$. Then time shifting is given as

$$\mathcal{L}\left[x(t-a)u(t-a)\right] = e^{-as}\,X(s) \tag{3.18}$$

If a function is delayed in time by a, then the Laplace transform of the function is multiplied by e^{-as}. The time shifting property can be proved as follows:

$$\mathcal{L}\left[x(t-a)u(t-a)\right] = \int_0^\infty x(t-a)u(t-a)e^{-st}\,dt, \quad a \geq 0 \tag{3.19}$$

If $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t > a$, Then,

$$\mathcal{L}\left[x(t-a)u(t-a)\right] = \int_0^a 0\,e^{-st}\,dt + \int_a^\infty x(t-a)(1)e^{-st}\,dt \tag{3.20}$$

By substituting $\lambda = t-a$, $d\lambda = dt$ and $t = \lambda + a$. As $t \to a$, $\lambda \to 0$ and as $t \to \infty$, $\lambda \to \infty$.

Thus,

$$\begin{aligned}\mathcal{L}\left[x(t-a)u(t-a)\right] &= \int_0^\infty x(\lambda)e^{-s(\lambda+a)}d\lambda \\ &= e^{-as}\int_0^\infty x(\lambda)e^{-s\lambda}d\lambda \\ &= e^{-as}X(s)\end{aligned} \tag{3.21}$$

For example, consider

$$g(t) = u(t-4) - u(t-6). \tag{3.22}$$

We know that $\mathcal{L}\left[u(t)\right] = \dfrac{1}{s}$

Using linearity property and the time shifting property,

$$G(s) = \mathcal{L}[u(t-4) - u(t-6)] = \frac{1}{s}e^{-4s} - \frac{1}{s}e^{-6s} = \frac{1}{s}\left(e^{-4s} - e^{-6s}\right) \tag{3.23}$$

3.3.4 Frequency Shifting

Let $X(s)$ be the Laplace transform of the signal $x(t)$. Then the frequency shifting is given as

$$\begin{aligned}\mathcal{L}\left[e^{-at}x(t)u(t)\right] &= \int_0^\infty e^{-at}x(t)e^{-st}\,dt = \int_0^\infty x(t)e^{-(s+a)t}\,dt = X(s+a)\\ \mathcal{L}\left[e^{-at}x(t)u(t)\right] &= X(s+a)\end{aligned} \tag{3.24}$$

The Laplace transform turns multiplication by e^{-at} in the time domain into $s + a$ in the transform $X(s)$.

For example, consider

$$\cos \omega t u(t) \Leftrightarrow \frac{s}{s^2+\omega^2} \tag{3.25}$$

By applying the frequency property and replacing every s by $s + a$, the Laplace transform of the cosine function is given by

$$\mathcal{L}[e^{-at}\cos \omega t u(t)] = \frac{s+a}{(s+a)^2+\omega^2} \tag{3.26}$$

3.3.5 Time Differentiation

The time differentiation property is important for solving differential equations. Let $X(s)$ be the Laplace transform of the signal $x(t)$. The Laplace transform of the derivative of $x(t)$ is given as

$$\mathcal{L}\left[\frac{dx}{dt}u(t)\right] = \int_{0^-}^\infty \frac{dx}{dt}e^{-st}\,dt \tag{3.27}$$

By differentiation by parts, $U = e^{-st}$, $dU = -se^{-st}\,dt$ and $dV = \left(\frac{dx}{dt}\right)dt = dx, V = x(t)$, then

$$\begin{aligned}\mathcal{L}\left[\frac{dx}{dt}u(t)\right] &= x(t)e^{-st}\Big|_{0^-}^\infty - \int_{0^-}^\infty x(t)\left[-se^{-st}\right]dt\\ &= 0 - x(0^-) + s\int_{0^-}^\infty x(t)e^{-st}dt\\ &= sX(s) - x\left(0^-\right)\\ \mathcal{L}\left[x'(t)\right] &= sX(s) - x\left(0^-\right)\end{aligned} \tag{3.28}$$

By differentiating the previous equation, we get

$$\begin{aligned}\mathcal{L}\left[\frac{d^2x}{dt^2}\right] &= s\mathcal{L}\left[x'(t)\right] - x'\left(0^-\right) = s\left[sX(s) - x\left(0^-\right)\right] - x'\left(0^-\right) \\ &= s^2X(s) - sx\left(0^-\right) - x'\left(0^-\right) \\ \mathcal{L}\left[x''(t)\right] &= s^2X(s) - sx\left(0^-\right) - x'\left(0^-\right)\end{aligned} \tag{3.29}$$

In the same way, for the *n*th derivative, we get

$$\mathcal{L}\left[\frac{d^nx}{dt^n}\right] = s^nX(s) - s^{n-1}x\left(0^-\right) - s^{n-2}x'\left(0^{-1}\right) - \cdots - s^0x^{(n-1)}(0^-) \tag{3.30}$$

For example, consider $x(t) = \cos \omega\, u(t)$. Then at $t = 0$, we have $x(0) = 1$. By applying time differentiation, the Laplace transform of the derivative of $x(t)$ is

$$\begin{aligned}\mathcal{L}\left[\frac{d}{dt}\cos\omega t u(t)\right] &= s\left(\frac{s}{s^2+\omega^2}\right) - \cos(0^-)u(0^-) \\ &= \frac{s^2}{s^2+\omega^2}\end{aligned} \tag{3.31}$$

3.3.6 Time Convolution

Let $X(s)$ and $H(s)$ be the Laplace transform of the signal $x(t)$ and $h(t)$. Then the convolution property is given as

$$\mathcal{L}\left[x(t) * h(t)\right] = X(s)H(s) \tag{3.32}$$

In the time domain, the convolution property is

$$x(t) * h(t) = \int_0^\infty x(t-\tau)h(\tau)d\tau$$

Considering the Laplace transform on both sides:

$$\begin{aligned}\mathcal{L}\left[x(t) * h(t)\right] &= \int_0^\infty \left[\int_0^\infty h(\tau)x(t-\tau)d\tau\right]e^{-st}\,dt \\ \mathcal{L}\left[x(t) * h(t)\right] &= \int_0^\infty h(\tau)\left[\int_0^\infty x(t-\tau)e^{-st}\,dt\right]d\tau\end{aligned} \tag{3.33}$$

Let $\lambda = t - \lambda$ so that $t = \tau + \lambda$ and $dt = d\lambda$

$$\begin{aligned}\mathcal{L}\left[x(t)*h(t)\right] &= \int_0^\infty h(\tau)\left[\int_0^\infty x(\lambda)e^{-s(t+\lambda)}\,d\lambda\right]d\tau \\ &= \int_0^\infty h(\tau)e^{-st}\,d\tau\int_0^\infty x(\lambda)e^{-s\lambda}\,d\lambda = H(s)X(s)\end{aligned} \tag{3.34}$$

For example, consider

$$g(t) = u(t-3) - u(t-4).$$

By applying the Laplace transform on both sides:

$$G(s) = \frac{1}{s}\left(e^{-3s} - e^{-4s}\right)$$

Considering the convolution of $g(t)*g(t)$, which is $G^2(s)$,

$$G^2(s) = \frac{1}{s^2}\left(e^{-3s} - e^{-4s}\right)^2 = \frac{1}{s^2}\left(e^{-6s} - 2e^{-7s} + e^{-8s}\right) \tag{3.35}$$

By applying the inverse Laplace transform, we get

$$\begin{aligned}x(t) = g(t)*g(t) &= (t-6)u(t-6) - 2(t-7)u(t-7) + (t-8)u(t-8) \\ &= r(t-6) - 2r(t-7) + r(t-8)\end{aligned} \tag{3.36}$$

3.3.7 Time Integration

Let $X(s)$ be the Laplace transform of the signal $x(t)$. Then Laplace transform of the integral is given as

$$\mathcal{L}\left[\int_0^t x(t)\,dt\right] = \frac{1}{s}X(s)$$

From the convolution property, we get

$$\begin{aligned}&\mathcal{L}\left[\int_0^t x(\lambda)\,d\lambda\right] = \mathcal{L}\left[\int_0^t x(\lambda)u(t-\lambda)\,d\lambda\right] = \mathcal{L}\left[u(t)*x(t)\right] = \frac{1}{s}X(s) \\ &\mathcal{L}\left[\int_0^t x(t)\,dt\right] = \frac{1}{s}X(s)\end{aligned} \tag{3.37}$$

For example, let $x(t) = u(t)$. We know that $X(s) = 1/s$. By applying the time integration property,

$$\mathcal{L}\left[\int_0^t u(t)dt\right] = \mathcal{L}[t] = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

3.3.8 Frequency Differentiation

Let $X(s)$ be the Laplace transform of the signal $x(t)$. Then the Laplace transform of the frequency differentiation is given as

$$\mathcal{L}\left[tx(t)\right] = -\frac{dX(s)}{ds}$$

Here, $X(s)$ is given as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

Differentiating both sides, we get

$$\frac{dX(s)}{ds} = \int_{0^-}^{\infty} x(t)(-te^{-st})\,dt = \int_{0^-}^{\infty} \left(-tx(t)\right)e^{-st}\,dt = \mathcal{L}\left[-tx(t)\right]$$

The frequency differentiation property is given as

$$\mathcal{L}\left[tx(t)\right] = -\frac{dX(s)}{ds} \tag{3.38}$$

Similarly, the general form of the frequency differentiation property is given as

$$\mathcal{L}\left[t^n x(t)\right] = (-1)^n \frac{d^n X(s)}{ds^n} \tag{3.39}$$

For example,

$$\mathcal{L}(t\cos\omega t) = -\frac{d}{ds}\left[\frac{s}{s^2+\omega^2}\right] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \tag{3.40}$$

$$\mathcal{L}(t\sin\omega t) = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2} \tag{3.41}$$

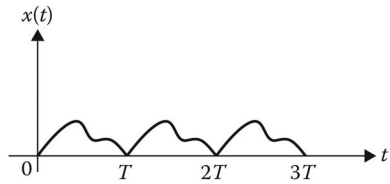


FIGURE 3.3 A periodic function.

3.3.9 Time Periodicity

Consider a periodic signal $x(t)$, and it can be expressed as a sum of time-shifted functions as (Figures 3.3 and 3.4)

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) + x_3(t) + \cdots \\ &= x_1(t) + x_1(t-T)u(t-T) + x_1(t-2T)u(t-2T) + \cdots \end{aligned} \tag{3.42}$$

where $0 < t < T$. By applying the Laplace transform and the time-shifting property, we get

$$\begin{aligned} X(s) &= X_1(s) + X_1(s)e^{-Ts} + X_1(s)e^{-2Ts} + X_1(s)e^{-3Ts} + L \\ &= X_1(s)\left[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + L\right] \end{aligned} \tag{3.43}$$

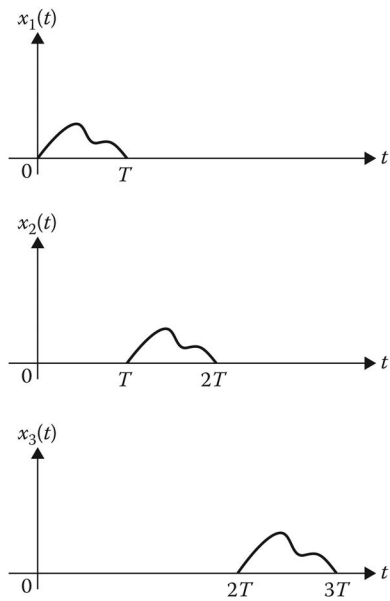


FIGURE 3.4 Decomposition of the periodic function in Figure 3.3.

But

$$1 + a + a^2 + a^3 + L = \frac{1}{1-a} \tag{3.44}$$

if $|a| < 1$.

$$X(s) = \frac{X_1(s)}{1 - e^{-Ts}} \tag{3.45}$$

For example, consider

$$x_1(t) = \begin{cases} \sin \pi t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \tag{3.46}$$

$$X_1(s) = \int_0^\infty x_1(t)\, e^{-st}\, dt = \int_0^1 \sin \pi t\, e^{-st}\, dt$$

$$\int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2 + b^2}(a \sin bt - b \cos bt)$$

where $a = -s$, $b = \pi$.

$$X_1(s) = \frac{e^{-st}}{s^2 + \pi^2}[-s \sin \pi t - \pi \cos \pi t]\Big|_0^1 = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}, \quad T = 2 \tag{3.47}$$

$$\begin{aligned} X(s) &= \frac{X_1(s)}{1 - e^{-Ts}} \\ &= \frac{\pi(1 + e^{-s})}{(s^2 + \pi^2)(1 - e^{-2s})} \end{aligned} \tag{3.48}$$

3.3.10 Modulation

Let $X(s)$ be the Laplace transform of the signal $x(t)$; then for any real number ω,

$$\begin{aligned} \mathcal{L}[x(t) cos\omega t] &= \frac{1}{2}[X(s + j\omega) + X(s - j\omega)] \\ \mathcal{L}[x(t) sin\omega t] &= \frac{j}{2}[X(s + j\omega) - X(s - j\omega)] \end{aligned} \tag{3.49}$$

The signals $x(t)cos\omega t$ and $x(t)sin\omega t$ can be expressed as

$$\begin{aligned} x(t) cos\omega t &= \frac{1}{2} x(t)\left[e^{-j\omega t} + e^{j\omega t}\right] \\ x(t) sin\omega t &= \frac{1}{2j} x(t)\left[e^{-j\omega t} - e^{j\omega t}\right] \end{aligned} \tag{3.50}$$

From eq. (3.24),

$$\mathcal{L}\left[e^{\pm j\omega t}x(t)\right]=X\left(s\pm j\omega\right) \tag{3.51}$$

For example, we know that

$$u(t)\Leftrightarrow\frac{1}{s}$$

$$u(t)\cos(\omega_o t)\Leftrightarrow\frac{1}{2}\left[\frac{1}{s+j\omega_o}+\frac{1}{s-j\omega_o}\right]=\frac{s}{s^2+\omega_o^2} \tag{3.52}$$

$$u(t)\sin(\omega_o t)\Leftrightarrow\frac{1}{2j}\left[\frac{1}{s+j\omega_o}-\frac{1}{s-j\omega_o}\right]=\frac{\omega_o}{s^2+\omega_o^2} \tag{3.53}$$

3.3.11 Initial and Final Values

The initial value $x(0)$ and final value $x(\infty)$ are used to relate frequency domain expressions to the time domain as time approaches zero and infinity, respectively. These properties can be derived by using the differentiation property as

$$sX(s)-x(0^-)=\mathcal{L}\left[\frac{dx}{dt}\right]=\int_{0^-}^{\infty}\frac{dx}{dt}e^{-st}\,dt \tag{3.54}$$

By taking limits, we get

$$\lim_{s\to\infty}\left[sX(s)-x(0)\right]=0$$

Since $x(0)$ is not dependent on s,

$$x(0)=\lim_{s\to\infty}sX(s) \tag{3.55}$$

This is called the initial-value theorem. $x(0)$ can be found using $X(s)$, without using the inverse transform of $x(t)$.

If we consider $s\to 0$, then

$$\begin{aligned}sX(s)-x\left(0^-\right)&=\int_{0^-}^{\infty}\frac{dx}{dt}e^{-0t}\,dt=\int_{0^-}^{\infty}dx=x(\infty)-x\left(0^-\right)\\ x(\infty)&=\lim_{s\to 0}sX(s)\end{aligned} \tag{3.56}$$

This is known as the final-value theorem. In the final-value theorem, all poles of $X(s)$ must be located in the left half of the s-plane.

For example, consider

$$x(t)=4+e^{-2t}\cos 3t\,u(t)\Leftrightarrow X(s)=\frac{4}{s}+\frac{s+2}{(s+2)^2+3^2} \tag{3.57}$$

By applying the initial-value theorem,

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} 4 + \frac{s^2+2s}{s^2+4s+13}$$

$$= \lim_{s\to\infty} 4 + \frac{1+\frac{2}{s}}{1+\frac{4}{s}+\frac{13}{s^2}} = 4 + \frac{1+0}{1+0+0} \tag{3.58}$$

$$x(0) = 5$$

As another example, consider

$$x(t) = (5+e^{-2t})\,u(t) \Leftrightarrow X(s) = \frac{5}{s} + \frac{1}{s+2} \tag{3.59}$$

Using the final-value theorem,

$$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\left(5 + \frac{s}{s+2}\right) = 5 + \frac{0}{0+2} = 5 \tag{3.60}$$

The final-value theorem cannot be used to find the value of $x(t) = cos2t$, because $X(s)$ has poles at $s = \pm j\sqrt{2}$, which are not in the left half of the s-plane.

For convenience, Table 3.1 provides the derived list of properties of the Laplace transform, while Table 3.2 gives the Laplace transform of some common functions. For signals in Table 3.2, we have omitted the factor $u(t)$ except where it is necessary because we are only interested in functions that exist for $t > 0$.

Example 3.3

Obtain the Laplace transform of

$$x(t) = \delta(t) - 3u(t) + 5e^{-2t}u(t).$$

SOLUTION

By the linearity property,

$$X(s) = \mathcal{L}[\delta(t)] - 3\mathcal{L}[u(t)] + 5\mathcal{L}[e^{-2t}u(t)]$$

$$= 1 - 3\frac{1}{s} + 5\frac{1}{s+2} = \frac{s^3+4s-6}{s(s+2)}$$

Practice Problem 3.3 Find the Laplace transform of $x(t) = [\cos 3t + e^{-5t}]u(t)$.

Answer: $X(s) = \dfrac{2s^2+5s+9}{(s+5)(s^2+9)}$.

TABLE 3.1
Properties of the Laplace Transform

Property	$x(t)$	$X(s)$
Linearity	$a_1x_1(t)+a_1x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Scaling	$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right), \quad a>0$
Time shifting	$x(t-a)u(t-a)$	$e^{-as}X(s)$
Frequency shifting	$e^{-at}x(t)$	$X(s+a)$
Time differentiation	$\frac{dx}{dt}$	$sX(s)-x(0^-)$
(second derivative)	$\frac{d^2x}{dt^2}$	$s^2X(s)-sx(0^{-1})-x'(0)$
(third derivative)	$\frac{d^3x}{dt^3}$	$s^3X(s)-s^2x(0^-)-sx'(0^-)-x''(0^-)$
(nth derivative)	$\frac{d^nx}{dt^n}$	$s^nX(s)-s^{n-1}x(0^-)-s^{n-2}x'(0^-)-\cdots-x^{(n-1)}(0^-)$
Time convolution	$x(t)*h(t)$	$X(s)H(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s)*X_2(s)$
Time integration	$\int_0^t x(t)dt$	$\frac{1}{s}X(s)$
Frequency integration	$\frac{x(t)}{t}$	$\int_s^\infty X(\lambda)d\lambda$
Frequency		
Differentiation	$tx(t)$	$-\frac{d}{ds}X(s)$
Multiplication by	$t^n\,t^nx(t)$	$(-1)^n\frac{d^n}{dx^n}X(s)$
Time periodicity	$x(t)=x(t+nT)$	$\frac{X_1(s)}{1-e^{-sT}}$
Modulation:		
(multiplication by cosωt)	$x(t)\,cos\omega t$	$\frac{1}{2}\left[X(s+j\omega)+X(s-j\omega)\right]$
(multiplication by sinωt)	$x(t)\,sin\omega t$	$\frac{j}{2}\left[X(s+j\omega)-X(s-j\omega)\right]$
Initial value	$x(0)$	$\lim_{s\to\infty}sX(s)$
Final value	$x(\infty)$	$\lim_{s\to0}sX(s)$

TABLE 3.2
Laplace Transform Pairs*

Entry no.	$x(t)$	$X(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$1/s$
3.	$u(t) - u(t-a)$	$\frac{1-e^{-as}}{s}$
4.	e^{-at}	$\frac{1}{s+a}$
5.	t	$\frac{1}{s^2}$
6.	t^n	$\frac{n!}{s^{n+1}}$
7.	te^{-at}	$\frac{1}{(s+a)^2}$
8.	$t^n\, e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
9.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
10.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
11.	$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
12.	$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
13.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
14.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
15.	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
16.	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
17.	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
18.	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$

* Defined for $t \geq 0$; $x(t) = 0$ for $t < 0$.

Example 3.4

Determine the Laplace transform of $x(t) = t^2u(t-1)$.

SOLUTION

We can rewrite $x(t)$ as

$$x(t) = (t-1+1)^2u(t-1) = (t-1)^2u(t-1) - 2(t-1)u(t-1) + u(t-1)$$

Using the time-shifting property, we obtain

$$X(s) = e^{-s}\left(\frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s}\right)$$

Practice Problem 3.4 Find the Laplace transform of $x(t) = e^{-3t}\cos 2tu(t)$

Answer: $\dfrac{s+3}{(s+3)^2+4}$.

Example 3.5

Determine the transform of the signal in Figure 3.5.

SOLUTION

We will solve this in three ways.

Method 1: Using the definition of the Laplace transform.

$$X(s) = \int_0^t x(t)e^{-st}dt = \int_1^2 2e^{-st}dt + \int_2^5 (-2)e^{-st}dt = 2\frac{e^{-st}}{-s}\Big|_0^2 - \frac{2e^{-st}}{-s}\Big|_2^3$$

$$= -\frac{2}{s}(e^{-2s} - e^{-s}) + \frac{2}{s}(e^{-3s} - e^{-2s}) = \frac{2}{s}\left[e^{-s} - 2e^{-2s} + e^{-3s}\right]$$

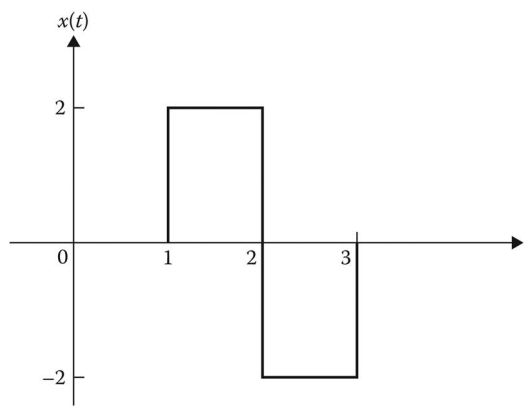


FIGURE 3.5 For Example 3.5.

Method 2: Expressing $x(t)$ in terms of unit step functions.

$$\begin{aligned} x(t) &= 2\big[u(t-1)-u(t-2)\big]-2\big[u(t-2)-u(t-3)\big] \\ &= 2u(t-1)-4u(t-2)+2u(t-3) \end{aligned}$$

Taking the Laplace transform of each term,

$$\begin{aligned} \mathrm{X(s)} &= \frac{2}{s}e^{-s}-\frac{4}{s}e^{-2s}+\frac{2}{s}e^{-3s} \\ &= \frac{2}{s}\big[e^{-s}-2e^{-2s}+e^{-3s}\big] \end{aligned}$$

Method 3: Taking the derivative of $x(t)$.

$$x'(t) = 2\delta(t-1)-4\delta(t-2)+2\delta(t-3)$$

Take the Laplace transfer of each term.

$$\mathrm{sX(s)} = 2e^{-s}-4e^{-2s}+2e^{-3s}$$

or

$$X(s) = \frac{2}{s}\big[e^{-s}-2e^{-2s}+e^{-3s}\big]$$

Practice Problem 3.5 Determine the Laplace transform of:

a. $x(t) = u(t)-u(t-2)$
b. $h(t)$ shown in Figure 3.6.

Answer: (a) $X(s) = \dfrac{(1-e^{-2s})}{s}$, (b) $H(s) = \dfrac{1}{s^2}-\dfrac{e^{-s}}{s^2}-\dfrac{e^{-s}}{s}$.

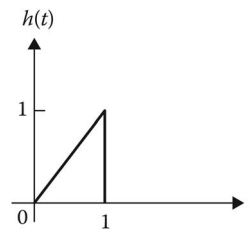


FIGURE 3.6 For Practice Problem 3.5(b).

Example 3.6

Let the Laplace transform of a signal $x(t)$ be

$$X(s) = \frac{s+2}{s^2+3s+5}$$

Determine the Laplace transform of the following signals:

a. $y(t) = 3x(t/2)$
b. $z(t) = tx(t-1)$
c. $h(t) = 3\dfrac{dx}{dt}$
d. $g(t) = \displaystyle\int_0^t x(\tau)d\tau$

SOLUTION

a. We apply the scaling property with $a = 1/2$.

$$Y(s) = 3\big(2X(2s)\big) = \frac{6(2s+2)}{(2s)^2+3(2s)+5} = \frac{12(s+1)}{4s^2+6s+5}$$

b. $z(t) = tx(t-1) = (t-1)x(t-1) + x(t-1)$
Taking the Laplace transform of each term and using the shifting property, we obtain

$$Z(s) = -e^{-s}\frac{d}{ds}X(s) + e^{-s}X(s) = -e^{-s}\frac{\big[(s^2+3s+5)(1)-(s+2)(2s+3)\big]}{(s^2+3s+5)^2} + e^{-s}X(s)$$

$$= \frac{(s^2+4s+1)e^{-s}}{(s^2+3s+5)^2} + \frac{(s+2)e^{-s}}{(s^2+3s+5)}$$

c. Applying the differentiation property,

$$H(s) = 3\big[sX(s) - x(0)\big]$$

The initial-value theorem gives

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\frac{s^2+2s}{s^2+3s+5} = \lim_{s\to\infty}\frac{1+2/s}{1+3/s+5/s^2} = 1$$

$$H(s) = 3\left[\frac{s^2+2s}{s^2+3s+5} - 1\right] = \frac{-3(s+5)}{s^2+3s+5}$$

d. Using the integration property,

$$G(s) = \frac{1}{s}X(s) = \frac{s+2}{s(s^2+3s+5)}$$

Practice Problem 3.6 Let $H(s) = \mathcal{L}[h(t)] = \dfrac{s+2}{s(s+5)}$. Find the Laplace transform of:

(a) $h(t-2)$, (b) $h(4t)$, (c) $\dfrac{d}{dt}h(t)$, (d) $h(t)*h(t)$

Answer: (a) $\dfrac{(s+2)e^{-2s}}{s(s+5)}$, (b) $\dfrac{s+8}{s(s+20)}$, (c) $\dfrac{-3}{s+5}$, (d) $\dfrac{(s+2)^2}{s^2(s+5)^2}$.

Example 3.7

Determine the initial and final values of $x(t)$, if they exist, given that:

$$X(s) = \frac{s^2 - 2s + 1}{(s-2)(s^2 + 2s + 4)}$$

SOLUTION

$$x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s^3 - 2s^2 + s}{(s-2)(s^2+2s+4)} = \lim_{s\to\infty} \frac{1 - \dfrac{2}{s} + \dfrac{1}{s^2}}{\left(1 - \dfrac{2}{s}\right)\left(1 + \dfrac{2}{s} + \dfrac{4}{s^2}\right)} = 1$$

One pole ($s = 2$) is not in the left-half plane. $x(\infty)$ does not exist.

Practice Problem 3.7 Find the initial and final values of

$$X(s) = \frac{s^2 + 3}{s^3 + 4s^2 + 6}$$

Answer: $x(0) = 1$; $x(\infty)$ does not exist.

3.4 THE INVERSE LAPLACE TRANSFORM

For a given $X(s)$, we can obtain the corresponding $x(t)$ by using eq. (3.4), but this requires contour integration, which is beyond the scope of this text. We will use an easier approach by employing the partial fraction expansion of $X(s)$ and matching entries in Table 3.2.

In most cases, we can express $X(s)$ in the standard form of

$$X(s) = \frac{N(s)}{D(s)} \tag{3.61}$$

where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial. Equation (3.61) can be expressed as

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0} \tag{3.62}$$

We can express $X(s)$ in terms of factors of $N(s)$ and $D(s)$ as:

$$X(s) = \frac{N(s)}{D(s)} = k\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{3.63}$$

where $k = b_m/a_n$. The roots of $N(s) = 0$ are called the *zeros* of $X(s)$; i.e., the values $-z_1, -z_2, \ldots, z_m$ are the zeros of $X(s)$.

Similarly, the roots of $D(s) = 0$ are the *poles* of $X(s)$; i.e., the values $-p_1, -p_2, \ldots, p_n$ are the poles of $X(s)$. We assume that $X(s)$ is proper ($m < n$); i.e., the degree of $N(s)$ is less than the degree of $D(s)$. [In case $X(s)$ is not proper ($m > n$), we use long division to reduce it to the proper fraction.] We use the partial fraction to break $X(s)$ down into simple terms whose inverse transform is obtained by matching entries in Table 3.2. Partial fraction expansions are useful in finding the inverse Laplace, inverse Fourier, or inverse z-transforms. Thus, to find the inverse Laplace transform of $X(s)$ requires two steps.

1. Use partial fraction expansion to break $X(s)$ into simple terms.
2. Refer to Table 3.2 to obtain the inverse Laplace transform of each term.

We will now consider the three common forms $X(s)$ may take and how we apply the two steps to each form.

3.4.1 Simple Poles

A simple pole is a first-order pole. If the poles $s = -p_1, -p_2, ..., -p_n$ are simple and distinct ($p_i \neq p_j$ for all $i \neq j$), then $D(s)$ in eq. (3.63) becomes a product of factors, so that

$$X(s) = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{3.64}$$

Using partial fraction expansion, we decompose $X(s)$ in eq. (3.64) as

$$X(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n} \tag{3.65}$$

To get the expansion coefficients $k_1, k_2, \ldots, k_n$ [known as the *residues* of $X(s)$], we multiply both sides of eq. (3.65) by $(s + pi)$ for each pole. For example, if we multiply by $(s + p_1)$, we obtain

$$(s+p_1)X(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \cdots + \frac{(s+p_1)k_n}{s+p_n} \tag{3.66}$$

Setting $s = -p_1$ in eq. (3.66) leaves only k_1 on the right-hand side of eq. (3.66). By isolating k_1 this way, we obtain

$$(s+p_1)X(s)\big|_{s=-p_1} = k_1$$

In general,

$$k_i = (s+p_i)X(s)\big|_{s=-p_i} \tag{3.67}$$

This is referred to as *Heaviside's formula*. From Table 3.2, the inverse transform of each term in eq. (3.65) is

$$\mathcal{L}^{-1}[k/(s+a)] = ke^{-at}u(t) \tag{3.68}$$

Applying eq. (3.68) to each term in eq. (3.65) leads to the inverse Laplace transform of $X(s)$ as

$$x(t) = \left(k_1e^{-p_1t} + k_2e^{-p_2t} + \cdots + k_ne^{-p_nt}\right)u(t) \tag{3.69}$$

3.4.2 Repeated Poles

If $X(s)$ has repeated poles, we employ a different strategy. Let $X(s)$ have n repeated poles at $s = -p$, then we may represent $X(s)$ as

$$X(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + X_1(s) \tag{3.70}$$

where $X_1(s)$ stands for the remaining part of $X(s)$ that does not contain the repeated poles. We find the coefficient kn as

$$k_n = (s+p)^n X(s)\big|_{s=-p} \tag{3.71}$$

as in eq. (3.68) To determine $kn - 1$, we multiply each term in eq. (3.71) by $(s+p)^n$, differentiate, and then set $s = -p$ to get rid of the other coefficients except $kn - 1$. Thus, we obtain

$$k_{n-1} = \frac{d}{ds}\left[(s+p)^n X(s)\right]\Big|_{s=-p} \tag{3.72}$$

We repeat the same process to get

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+p)^n X(s)\right]\Big|_{s=-p} \tag{3.73}$$

The mth repetition gives

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}\left[(s+p)^n X(s)\right]\Big|_{s=-p} \tag{3.74}$$

where $m = 1, 2, 3,\ldots, n-1$. From Table 3.2,

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!} \tag{3.75}$$

We apply this to each term in eq. (3.70) and obtain

$$x(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^3 e^{-pt} + \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + x_1(t) \tag{3.76}$$

3.4.3 Complex Poles

We now consider when $X(s)$ has complex poles. Complex poles occur in complex conjugate pairs so that the number of poles is even. A pair of complex poles is simple if it is not repeated. Simple complex poles may be handled the same way as simple real poles, but the procedure involves complex algebra, which can be intimidating. An easier approach involves *completing the squares*. The idea is to express each complex pole pair (or quadratic term) in $D(s)$ as a complete square such as $(s+\alpha)^2 + \beta^2$ and then use Table 3.2 to find the inverse of the term.

For the sake of simplicity, we will assume that $X(s)$ has only one complex pole pair. If $X(s)$ has complex roots, its denominator $D(s)$ will contain a quadratic term and $X(s)$ may have the form

$$X(s) = \frac{A_1 s + A_2}{s^2 + as + b} + X_1(s) \tag{3.77}$$

where $X_1(s)$ stands for the remaining part of $X(s)$ that does not contain this pair of complex poles. We complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s+\alpha)^2 + \beta^2 \tag{3.78}$$

and we also let

$$A_1 s + A_2 = A_1(s+\alpha) + B_1\beta$$

Thus, eq. (3.77) becomes

$$X(s) = \frac{A_1(s+\alpha)}{(s+\alpha)^2 + \beta^2} + \frac{B_1\beta}{(s+\alpha)^2 + \beta^2} + X_1(s) \tag{3.79}$$

Using entries in Table 3.2, we obtain the inverse Laplace transform of each term in eq. (3.79) so that

$$x(t) = \left(A_1 e^{-\alpha t}\cos\beta t + B_1 e^{-\alpha t}\sin\beta t\right)u(t) + x_1(t) \tag{3.80}$$

The sine and cosine terms can be combined if desired. We use

$$A\cos\theta + B\sin\theta = R\cos(\theta - \phi) \tag{3.81}$$

where $R = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}(B/A)$.

In addition to the three common forms considered earlier, we can have two other cases. First, the term may involve e^{-sT}. For example, $X(s)e^{-sT}$ corresponds to $x(t - T)u(t - T)$ using the time shifting property. Second, for $X(s) = N(s)/D(s)$, when the degree of the numerator N(s) is higher than the degree of the denominator D(s), we use long division to express $X(s)$. These two cases are best illustrated with Examples 3.13 and 3.14.

Example 3.8

Find the inverse Laplace transform of

$$X(s) = 1 + \frac{2}{s} + \frac{4}{s-1} - \frac{3s}{s^2+9}$$

SOLUTION

The inverse transform is obtained by consulting Table 3.2 for each term.

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}(1) + \mathcal{L}^{-1}\left(\frac{2}{s}\right) + \mathcal{L}^{-1}\left(\frac{4}{s-1}\right) - 3\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) \\ &= \delta(t) + (2 + 4e^{t} - 3\cos 3t)u(t) \end{aligned}$$

Practice Problem 3.8 Determine the inverse Laplace transform of

$$X(s) = \frac{1}{s^2} - \frac{2}{s+3} + \frac{4}{s^2+4}$$

Answer: $(t - 2e^{-3t} + 2\sin 2t)u(t)$.

Example 3.9

Obtain $h(t)$ given that

$$H(s) = \frac{4}{(s+1)(s+3)}$$

SOLUTION

In the previous example, the partial fractions were provided. Here we determine the partial fractions. We let

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} \tag{3.9.1}$$

where A and B are constants. The constants can be determined in two ways.

Method 1: Residue method

$$A = (s+1)H(s)\Big|_{s=-1} = \frac{4}{(s+3)}\Big|_{s=-1} = \frac{4}{(2)} = 2$$

$$B = (s+3)F(s)\Big|_{s=-3} = \frac{4}{(s+1)}\Big|_{s=-3} = \frac{4}{(-2)} = -2$$

Method 2: Algebraic method

Multiplying both sides of eq. (3.9.1) by $(s + 1)(s + 3)$ and equating the numerator, we get

$$4 = A(s + 3) + B(s + 1)$$

Equating the coefficients of like powers of s,

constant: $4 = 3A + B$
s: $0 = A + B \rightarrow B = -A$

Solving these gives $A = 2$, $B = -2$
Thus,

$$H(s) = \frac{4}{(s+1)(s+3)} = \frac{2}{s+1} - \frac{2}{s+3}$$

By taking the inverse transform for each term, we obtain

$$h(t) = \left(2e^{-t} - 2e^{-3t}\right)u(t)$$

Practice Problem 3.9 Find $x(t)$ if

$$X(s) = \frac{10s}{(s+1)(s+2)(s+3)}$$

Answer: $x(t) = (-5\,e^{-t} + 20\,e^{-2t} - 15\,e^{-3t})u(t)$

Example 3.10

Determine $x(t)$ given that

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3}$$

SOLUTION

The previous example is on simple roots, while this example is on repeated roots. Let

$$X(s) = \frac{2s^2 + 4s + 1}{(s+1)(s+2)^3} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} \tag{3.10.1}$$

We can determine the constants A, B, C, and D in two ways.

Method 1: Residue method

$$A = F(s)(s+1)\Big|_{s=-1} = \frac{2-4+1}{-1+2} = -1$$

$$D = F(s)(s+2)^3\Big|_{s=-2} = \frac{8-8+1}{-2+1} = -1$$

$$C = \frac{d}{ds}\Big[(s+2)^3 X(s)\Big]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{2s^2+4s+1}{(s+1)}\right]\Big|_{s=-2}$$

$$= \frac{(s+1)(4s+4) - (2s^2+4s+1)\times 1}{(s+1)^2}\Big|_{s=-2} = \frac{2s^2+4s+3}{(s+1)^2}\Big|_{s=-2}$$

$$= \frac{8-8+3}{1} = 3$$

$$B = \frac{1}{2}\frac{d^2}{ds^2}\Big[(s+2)^3 X(s)\Big]\Big|_{s=-2} = \frac{1}{2}\frac{d}{ds}\left[\frac{2s^2+4s+3}{(s+1)^2}\right]\Big|_{s=-2}$$

$$= \frac{(s+1)^2(4s+4) - (2s^2+4s+3)\times 2(s+1)}{2(s+1)^4}\Big|_{s=-2} = \frac{-2}{2(s+1)^3}\Big|_{s=-2}$$

$$= \frac{-2}{-2} = 1$$

Method 2: Algebraic method
Multiplying eq. (3.10.1) by $(s+1)(s+2)^3$ and equating the numerator, we obtain

$$2s^2 + 4s + 1 = A(s+2)^3 + B(s+1)(s+2)^2 + C(s+1)(s+2) + D(s+1)$$

or

$$2s^2 + 4s + 1 = A(s^3 + 6s^2 + 12s + 8) + B(s^3 + 5s^2 + 8s + 4) + C(s^2 + 3s + 2) + D(s+1)$$

Equating coefficients,

$$\begin{aligned} s^3&: 0 = A + B \rightarrow B = -A \\ s^2&: 2 = 6A + 5B + C \rightarrow 2 = A + C \\ s&: 4 = 12A + 8B + 3C + D \rightarrow 4 = 4A + 3C + D \\ \text{constant}&: 1 = 8A + 4B + 2C + D \rightarrow 1 = 4A + 2C + D \end{aligned}$$

Solving these simultaneously gives $A = -1$, $B = 1$, $C = 3$, $D = -1$ so that eq. (3.10.1) becomes

$$X(s) = \frac{-1}{s+1} + \frac{1}{s+2} + \frac{3}{(s+2)^2} - \frac{1}{(s+2)^3}$$

By taking the inverse transform of each term, we obtain

$$x(t) = \left(-e^{-t} + e^{-2t} + 3te^{-2t} - \frac{t^2}{2}e^{-2t}\right)u(t)$$

Practice Problem 3.10 Obtain $y(t)$ if

$$Y(s) = \frac{12}{(s+2)^2(s+4)}$$

Answer: $\left(3e^{-4t} - 3e^{-2t} + 6te^{-2t}\right)u(t)$.

Example 3.11

Find the inverse transform of

$$G(s) = \frac{s+1}{(s+3)(s^2+2s+5)}$$

SOLUTION

$G(s)$ has a pair of complex poles at $s^2 + 2s + 5 = 0$ or $s = -2 \pm j2$. We let

$$G(s) = \frac{s+1}{(s+2)(s^2+2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2s+5}$$

$$A = G(s)(s+2)\Big|_{s=-2} = \frac{-1}{5}$$

$$s+1 = A(s^2+2s+5) + B(s^2+2s) + C(s+2)$$

Equating coefficients:

$$s^2: 0 = A + B \rightarrow B = -A = \frac{1}{5}$$

$$s^1: 1 = 2A + 2B + C = 0 + C \rightarrow C = 1$$

$$s^0: 1 = 5A + 2C = -1 + 2 = 1$$

$$G(s) = \frac{-1/5}{s+2} + \frac{1/5 \cdot s + 1}{(s+1)^2 + 2^2} = \frac{-1/5}{s+2} + \frac{1/5(s+1)}{(s+1)^2 + 2^2} + \frac{4/5}{(s+1)^2 + 2^2}$$

$$g(t) = (-0.2e^{-2t} + 0.2e^{-t}\cos(2t) + 0.4e^{-t}\sin(2t))u(t)$$

Practice Problem 3.11 Find $v(t)$ given that

$$V(s) = \frac{2s + 26}{s(s^2 + 4s + 13)}$$

Answer: $v(t) = (2 - 2e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t)u(t)$.

Example 3.12

Consider the rectangular pulse or gate function $x(t) = u(t) - u(t - 2)$. Obtain $y(t) = x(t)*x(t)$, i.e., the convolution of the rectangular pulse with itself.

SOLUTION

The Laplace transform of $x(t)$ is

$$X(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

Using the convolution property,

$$Y(s) = X(s)X(s) = X^2(s) = \frac{1}{s^2}(1 - 2e^{-2s} + e^{-4s})$$

Taking the inverse Laplace transform of each term,

$$\begin{aligned} y(t) &= x(t) * x(t) = tu(t) - 2(t-2)u(t-2) + (t-4)u(t-4) \\ &= r(t) - 2r(t-2) + r(t-4) \end{aligned}$$

which is a triangular pulse, as expected.

Practice Problem 3.12 Given the unit ramp function $r = tu(t)$ and the gate function $g(t) = u(t) - u(t - 1)$, use the convolution property to find $y(t) = r(t)*g(t)$.

Answer: $y(t) = 0.5t^2u(t) - 0.5(t-1)^2u(t-1)$.

Example 3.13

Determine the inverse Laplace transform of

$$X(s) = \frac{se^{-2s} + e^{-3s}}{s(s^2 + 5s + 4)}$$

SOLUTION

$$\text{Let } X(s) = X_1(s)e^{-2s} + X_2(s)e^{-3s} \tag{3.13.1}$$

where

$$X_1(s) = \frac{s}{s(s^2+5s+4)} = \frac{1}{(s+1)(s+4)} \tag{3.13.2}$$

$$X_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{s(s+1)(s+4)} \tag{3.13.3}$$

We now use partial fractions to obtain $x_1(t)$ and $x_2(t)$ from $X_1(s)$ and $X_2(s)$, respectively. Let

$$X_1(s) = \frac{1}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

where

$$A = \left.\frac{1}{s+4}\right|_{s=-1} = \frac{1}{-1+4} = \frac{1}{3}$$

$$B = \left.\frac{1}{s+1}\right|_{s=-4} = \frac{1}{-4+1} = -\frac{1}{3}$$

Hence,

$$X_1(s) = \frac{1}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right]$$

Taking the inverse of each term gives

$$x_1(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)u(t) \tag{3.13.4}$$

Similarly,

$$X_2(s) = \frac{1}{s(s+1)(s+4)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+4}$$

where

$$C = \left.\frac{1}{(s+1)(s+4)}\right|_{s=0} = \frac{1}{(1)(4)} = \frac{1}{4}$$

$$D = \left.\frac{1}{s(s+4)}\right|_{s=-1} = \frac{1}{(-1)(3)} = -\frac{1}{3}$$

$$E = \left.\frac{1}{s(s+1)}\right|_{s=-4} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

Thus,

$$X_2(s) = \frac{1}{s(s+1)(s+4)} = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}$$

From this, we obtain

$$x_2(t) = \left[\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right]u(t) \tag{3.13.5}$$

From eq. (3.13.1),

$$x(t) = x_1(t-2)u(t-2) + x_2(t-3)u(t-3) \tag{3.13.6}$$

Substituting eqs. (3.13.4) and (3.13.5) into eq. (3.13.6) finally gives.

$$x(t) = \frac{1}{3}\left(e^{-(t-2)} - e^{-4(t-2)}\right)u(t-2) + \left[\frac{1}{4} - \frac{1}{3}e^{-(t-3)} + \frac{1}{12}e^{-4(t-3)}\right]u(t-3)$$

Practice Problem 3.13 Determine the inverse of

$$Y(s) = \frac{1 - se^{-4s}}{(s+1)(s+2)}$$

Answer: $\left(e^{-t} - e^{-2t}\right)u(t) + \left(e^{-(t-4)} - 2e^{-2(t-4)}\right)u(t-4)$.

Example 3.14

Obtain $g(t)$ given that

$$G(s) = \frac{s^3 + 5s^2 + 10}{s^2 + 3s + 2}$$

SOLUTION

The degree of the numerator polynomial is greater than the degree of the denominator. In this case, the fraction is a proper one. The numerator is divided by the denominator by long division. We use long division until the remainder is one degree less than the denominator polynomial.

$$\begin{array}{r} s+2 \\ s^2+3s+2\overline{)\,s^3+5s^2+0s+10} \\ \underline{s^3+3s^2+2s} \\ 2s^2-2s+10 \\ \underline{2s^2+6s+4} \\ -8s+6 \end{array}$$

Thus,

$$G(s) = s + 2 + \frac{-8s + 6}{s^2 + 3s + 2}$$

Let $G(s) = s + 2 + Y(s)$, where $Y(s) = \dfrac{-8s + 6}{s^2 + 3s + 2}$.

Let $Y(s) = \dfrac{-8s + 6}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$,

where

$$A = \left.\frac{-8s + 6}{s + 2}\right|_{s=-1} = \frac{8 + 6}{-1 + 2} = 14$$

$$B = \left.\frac{-8s + 6}{s + 1}\right|_{s=-2} = \frac{16 + 6}{-2 + 1} = -22$$

$$Y(s) = \frac{14}{s+1} - \frac{22}{s+2}$$

Thus,

$$G(s) = s + 2 + \frac{14}{s+1} - \frac{22}{s+2}$$

Taking the inverse of each term gives

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + \left(14e^{-t} - 22e^{-2t}\right)u(t)$$

Practice Problem 3.14 Find $y(t)$ given that

$$Y(s) = \frac{s(s+1)}{(s-1)(s+2)}$$

Answer: $y(t) = \delta(t) + \dfrac{2}{3}\left(e^{t} - e^{-2t}\right)u(t)$.

3.5 TRANSFER FUNCTION

Now that we have mastered finding the Laplace transform and its inverse, we begin the application of the Laplace transform to the study of continuous-time systems. We begin with the concept of transfer functions.

We know from Chapter 2 that the output $y(t)$ of a continuous-time system is the convolution of the input $x(t)$ and the impulse response $h(t)$; i.e.,

$$y(t) = x(t) * h(t)$$

FIGURE 3.7 Impulse response $h(t)$ and transfer function $H(s)$.

By applying the convolution property, we obtain

$$Y(s) = X(s)\, H(s)$$

or

$$H(s) = \frac{Y(s)}{X(s)}$$

The Laplace $H(s)$ [which is the Laplace transform of $h(t)$] is known as the *transfer function* or system function of the system. The transfer function of a system describes how the output behaves with respect to the input.

> The **transfer function** $H(s)$ is defined as the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

Figure 3.7 shows the relationship between the impulse response $h(t)$ and transfer function $H(s)$. Given the system transfer function, we can perform extensive system analysis and design without having to apply specific signals to the systems.

A linear system is often specified in terms of some block diagram. Here we consider three types of interconnections.

Consider a parallel interconnection of two systems with transfer function $H_1(s)$ and $H_1(s)$ with single input $X(s)$ and single output $Y(s)$, as shown in Figure 3.8.

$$Y(s) = \left[H_1(s) + H_2(s)\right] X(s) \tag{3.82}$$

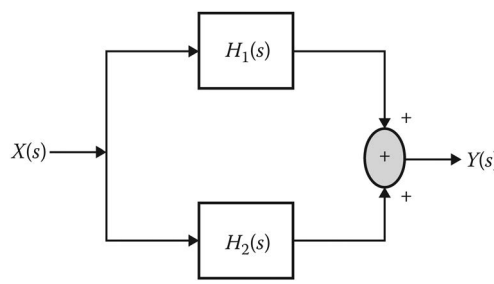


FIGURE 3.8 Parallel interconnection of two systems.

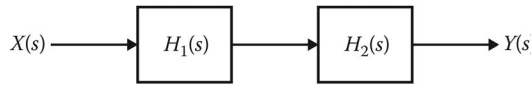


FIGURE 3.9 Series interconnection of two systems.

or

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) + H_2(s) \tag{3.83}$$

where $H(s)$ in the transfer function for the overall system.

We next consider a series interconnection (or cascade connection) of two systems with transfer function $H_1(s)$ and $H_1(s)$ with single input $X(s)$ and single output $Y(s)$, as shown in Figure 3.9.

$$Y(s) = H_1(s)H_2(s)X(s)$$

or

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s)H_2(s) \tag{3.84}$$

where $H(s)$ is the transfer function for the overall system.

Finally, we consider the feedback configuration in Figure 3.10. In this case, the output is feedback to the input. Negative feedback is common in practice because positive feedback can cause system instability. From Figure 3.10,

$$Y(s) = G(s)U(s) \tag{3.85}$$

But

$$U(s) = X(s) - F(s)Y(s) \tag{3.86}$$

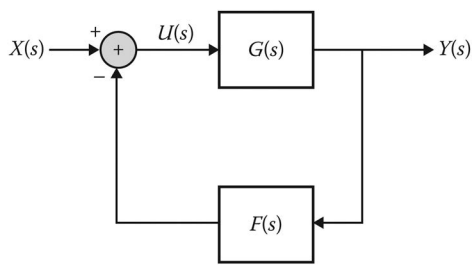


FIGURE 3.10 Feedback interconnection of systems.

Substituting eq. (3.86) into eq. (3.85) gives

$$Y(s) = G(s)\left(X(s) - F(s)Y(s)\right)$$

or

$$Y(s) + G(s)F(s)Y(s) = G(s)X(s) \tag{3.87}$$

Dividing both sides by $X(s)$ gives the transfer function for the overall system as

$$H(s) = \frac{G(s)}{1 + F(s)G(s)} \tag{3.88}$$

There are two special cases of this. First, if the subtractor in Figure 3.10 is an adder instead, then eq. (3.88) becomes

$$H(s) = \frac{G(s)}{1 - F(s)G(s)} \tag{3.89}$$

Second, for unity feedback $F(s) = 1$, and eq. (3.88) becomes

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)} \tag{3.90}$$

Example 3.15

The output of a linear system is $y(t) = e^{-t}\cos 4tu(t)$ when the input is $x(t) = 10e^{-t}u(t)$. Find the transfer function of the system.

SOLUTION

If $x(t) = e^{-t}\,u(t)$ and $y(t) = 10e^{-t}\cos 4t\,u(t)$, then

$$X(s) = \frac{1}{s+1} \quad \text{and} \quad Y(s) = \frac{10(s+1)}{(s+1)^2 + 4^2}$$

Hence

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10(s+1)^2}{(s+1)^2 + 16} = \frac{10(s^2 + 2s + 1)}{s^2 + 2s + 17}$$

To find $h(t)$, which is the inverse Laplace transform of $H(s)$, we write $H(s)$ as

$$H(s) = 10 - 40\frac{4}{s^2 + 2s + 17}$$

Taking the Laplace inverse of each and using Table 3.2, we obtain

$$h(t) = 10\delta(t) - 40\ e^{-t}\ \sin 4t\ u(t)$$

Practice Problem 3.15 The transfer function of a linear system is

$$H(s) = \frac{2s}{s+6}$$

Find the output $y(t)$ due to the input $e^{-3t}u(t)$ and its impulse response.

Answers: $(-2e^{-3t} + 4e^{-6t})u(t)$, $2\delta(t) - 12e^{-6t}u(t)$.

Example 3.16

Find the transfer function for the feedback system shown in Figure 3.11.

SOLUTION

We apply eq. (3.84) to obtain

$$E_2 = G_1G_2E_1$$
$$Y = G_3E_3$$

Applying eq. (3.82),

$$E_1 = X - F_2Y$$
$$E_3 = E_2 - F_1Y$$

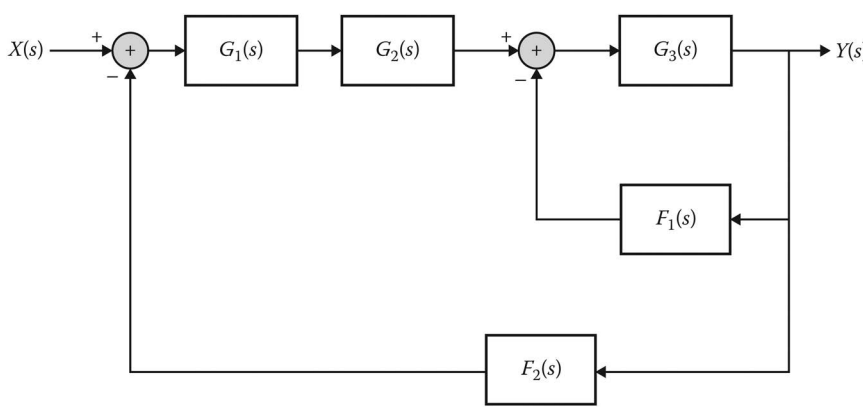


FIGURE 3.11 For Example 3.16.

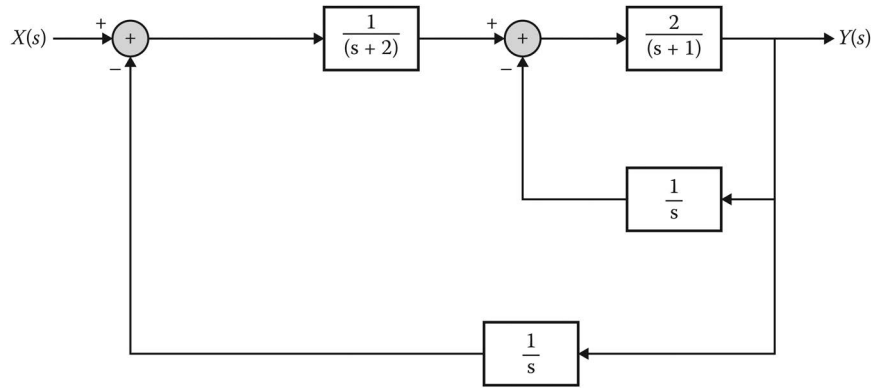


FIGURE 3.12 For Practice Problem 3.16.

If we eliminate E_1, E_2, and E_3, we get

$$\frac{Y}{G_3} = G_1G_2\left[X - F_2Y\right] - F_1Y$$

Rearranging things leads to the transfer function.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G_1(s)G_2(s)G_3(s)}{1 + F_1(s)G_3(s) + F_2(s)G_1(s)G_2(s)G_3(s)}$$

Practice Problem 3.16 Obtain the overall transfer function for the feedback system shown in Figure 3.12.

Answer: $H(s) = \dfrac{2s}{s^3 + 2s^2 + 3s + 3}$.

3.6 APPLICATIONS

Besides applying the Laplace transform to determine the transfer function of a system, the Laplace transform finds applications in many areas such as circuit analysis, control systems, communication systems, speech control, and radar systems. In this section, we will apply the Laplace transform to three areas: integrodifferential equations, circuit analysis, and control systems.

3.6.1 Integrodifferential Equations

We use the Laplace transform in solving linear integrodifferential equations with constant coefficients. This basically involves employing the differentiation and integration properties of the Laplace transform to the terms in the integrodifferential equation. Initial conditions are automatically incorporated. We solve the resulting algebraic

equation in the s-domain. Using the inverse Laplace transfer, we convert the solution back to the time domain. We illustrate the process with the following examples.

Example 3.17

Consider the system characterized by

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{-t}u(t)$$

Use the Laplace transform to solve the differential equation subject to

$$y(0) = 1, \quad \frac{dy(0)}{dt} = 0.$$

SOLUTION

Take the Laplace transform of each term.

$$\left[s^2Y(s) - sy(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + -4Y(s) = \frac{1}{s+1}$$

We substitute the initial conditions

$$s^2Y(s) - s - 0 + 3sY(s) - 3 - 4Y(s) = \frac{1}{s+1}$$

$$(s^2 + 3s - 4)Y(s) = s + 3 + \frac{1}{s+1} = \frac{s^2 + 4s + 4}{s+1}$$

$$Y(s) = \frac{s^2 + 4s + 4}{(s^2 + 3s - 4)(s+1)} = \frac{s^2 + 4s + 4}{(s-1)(s+1)(s+4)}$$

Using partial fractions, we let

$$Y(s) = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{(s+4)}$$

where

$$A = (s-1)Y(s)\Big|_{s=1} = \frac{1+4+4}{2(5)} = \frac{9}{10}$$

$$B = (s+1)Y(s)\Big|_{s=-1} = \frac{1-4+4}{(-2)(3)} = \frac{1}{-6}$$

$$C = (s+4)Y(s)\Big|_{s=-4} = \frac{16-16+4}{(-5)(-3)} = \frac{4}{15}$$

Thus,

$$Y(s) = \frac{9/10}{(s-1)} - \frac{1/6}{(s+1)} + \frac{4/15}{(s+4)}$$

Taking the inverse Laplace transform of each term,

$$y(t) = \left(\frac{9}{10}e^{t} - \frac{1}{6}e^{-t} + \frac{4}{15}e^{-4t}\right)u(t)$$

Practice Problem 3.17 A casual LTI system is described by

$$y''(t) + y'(t) + 2y(t) = x(t)$$

For system input $x(t) = 2u(t)$, find the output $y(t)$. Assume zero initial conditions.

Answer: $y(t) = (1 - 2e^{-t} + e^{-2t})u(t)$.

Example 3.18

Solve the integrodifferential equation

$$\frac{dv}{dt} + 2v + 5\int_{0}^{t} v(\lambda)d\lambda = 4u(t)$$

with $v(0) = -1$ and determine $v(t)$ for $t > 0$.

SOLUTION

Take the Laplace transform of each term of the integrodifferential equation. We take the Laplace transform of each term.

$$[sV(s) - v(0)] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s}$$

$$[sV(s) + 1] + 2V(s) + \frac{5}{s}V(s) = \frac{4}{s} \rightarrow V(s) = \frac{4 - s}{s^2 + 2s + 5}$$

$$V(s) = \frac{-(s+1) + 5}{(s+1)^2 + 2^2} = \frac{-(s+1)}{(s+1)^2 + 2^2} + 5/2\,\frac{2}{(s+1)^2 + 2^2}$$

Taking the inverse of each term leads to

$$v(t) = (-e^{-t}\cos 2t + 2.5e^{-t}\sin 2t)u(t)$$

Practice Problem 3.18 Use the Laplace transform to solve for $x(t)$ in

$$x(t) = \cos t + \int_{0}^{t} e^{\lambda - t}x(\lambda)d\lambda$$

Answer: $x(t) = \frac{1}{2}e^{t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t,\ t > 0.$

3.6.2 Circuit Analysis

The Laplace transform is a powerful tool in analyzing electric circuits. The use of the Laplace transform in circuit analysis facilitates the use of various signal sources such as impulse, step, ramp, exponential, and sinusoidal. Applying the Laplace transform to analyze circuits usually involves three steps:

1. Laplace-transform the circuit from the time domain to the frequency domain (or s-domain).
2. The circuit in the s-domain is solved using circuit analysis techniques (such as voltage division, current division, nodal analysis, mesh analysis, source transformation, and superposition), and we obtain the desired quantity $X(s)$.
3. Obtain the inverse Laplace transform of $X(s)$ to get the desired solution $x(t)$ in the time domain.

The three steps are illustrated in Figure 3.13 Steps 2 and 3 are familiar. Only the first step is new and will be discussed here. We transform a circuit in the time domain to frequency or s-domain by Laplace-transforming each element of the circuit.

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t) \tag{3.91}$$

Laplace-transforming this gives

$$V(s) = RI(s) \tag{3.92}$$

For the inductor,

$$v(t) = L\frac{di}{dt} \tag{3.93}$$

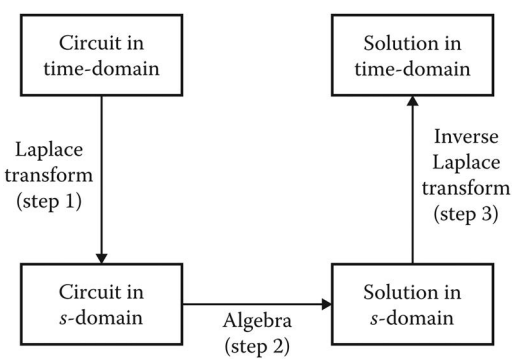


FIGURE 3.13 The three steps involved analyzing a circuit with Laplace transformation.

Laplace-transforming both sides yields

$$V(s) = L\left[sI(s) - i(0^-)\right] = sLI(s) - Li(0^-)$$

or

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s} \tag{3.94}$$

For the capacitor,

$$i(t) = C\frac{dv(t)}{dt} \tag{3.95}$$

This transforms into the s-domain as

$$I(s) = C\left[sV(0s) - v(0^-)\right] = sCV(s) - Cv(0^-)$$

or

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s} \tag{3.96}$$

Notice in eqs. (3.94) and (3.96) that the initial conditions are incorporated into the transformation process. This is one of the key advantages of using the Laplace transform in circuit analysis. Another advantage is that a complete response (natural and steady state) of a network is obtained at once.

Assuming zero initial conditions for the inductor and the capacitor, the equations reduce to:

$$\text{Resistor: } V(s) = I(s)R \tag{3.97a}$$

$$\text{Inductor: } V(s) = sL\,I(s) \tag{3.97b}$$

$$\text{Capacitor: } V(s) = \frac{1}{sC}I(s) \tag{3.97c}$$

Example 3.19

Find $v_o(t)$ in the circuit in Figure 3.14.

SOLUTION

We first transform the circuit from the time domain to the s-domain.

$$2u(t) \Rightarrow \frac{2}{s}$$
$$1\text{H} \Rightarrow sL = s$$
$$0.2\text{F} \Rightarrow \frac{1}{sC} = \frac{1}{0.2s} = \frac{5}{s}$$

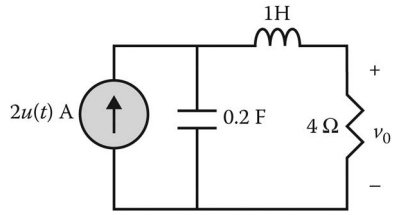


FIGURE 3.14 For Example 3.19.

Hence the frequency-domain equivalent circuit is in Figure 3.15. Using current division,

$$I_o = \frac{5/s}{5/s + s + 4} \times \frac{2}{s} = \frac{5}{s^2 + 4s + 5} \times \frac{2}{s}$$

$$V_o = 4I_o = \frac{40}{s(s^2 + 4s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 5}$$

Equating the denominators,

$$40 = A(s^2 + 4s + 5) + Bs^2 + Cs$$

Equating coefficients:

$$\text{constant: } 40 = 5A \rightarrow A = 8$$

$$s^1\text{: } 0 = 4A + C \rightarrow C = -4A = -32$$

$$s^2\text{: } 0 = A + B \rightarrow B = -A = -8$$

$$V_o = \frac{8}{s} - \frac{8s + 32}{(s+2)^2 + 1}$$

$$= \frac{8}{s} - \frac{8(s+2)}{(s+2)^2 + 1} - \frac{16}{(s+2)^2 + 1}$$

Taking the inverse Laplace transform of each term, we obtain

$$v_o(t) = \left(8 - 8e^{-2t}\cos t - 8e^{-2t}\sin t\right)u(t)$$

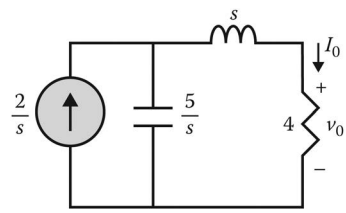


FIGURE 3.15 The *s*-domain circuit for Example 3.19.

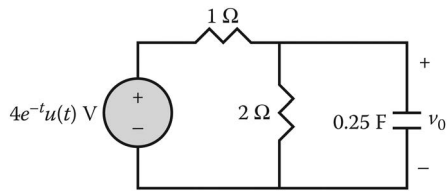


FIGURE 3.16 For Practice Problem 3.19.

Practice Problem 3.19 Using Laplace transformation, find $v_o(t)$ in the circuit in Figure 3.16.

Answer: $v_o(t) = 3.2\left(e^{-t} - e^{-6t}\right)u(t)$.

Example 3.20

For the circuit in Figure 3.17, find $H(s) = V_0(s)/V_s(s)$. Assume zero initial conditions.

SOLUTION

Transform the circuit to the s-domain as shown in Figure 3.18. Let

$$Z = \frac{1}{s} \| (s+2) = \frac{\frac{1}{s}(s+2)}{\frac{1}{s} + s + 2} = \frac{(s+2)}{(s^2 + 2s + 1)}$$

Using voltage division,

$$V_1 = \frac{Z}{Z+1} V_s$$

$$V_o = \frac{2}{s+2} V_1 = \frac{2}{s+2} \frac{Z}{Z+1} V_s$$

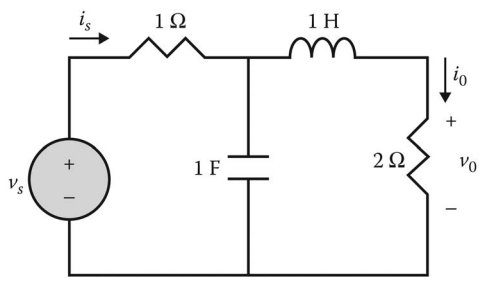


FIGURE 3.17 For Example 3.20 and Practice Problem 3.20.

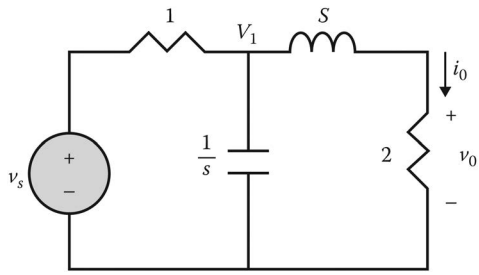


FIGURE 3.18 For Example 3.20.

$$H(s) = \frac{V_o}{V_s} = \frac{2}{s+2} \times \frac{\dfrac{s+2}{s^2+2s+1}}{\dfrac{s+2}{s^2+2s+1}+1}$$

$$H(s) = \frac{2(s+2)}{s^2+3s+3}$$

Practice Problem 3.20 For the circuit in Figure 3.17, obtain $H(s) = I_o(s)/I_s(s)$. Assume zero initial conditions.

Answer: $H(s) = \dfrac{1}{s^2+2s+1}$.

3.6.3 Control Systems

A major application of the Laplace transform is found in the study of control systems. An important consideration of control systems is their stability. We examine the bounded-input bounded-output (BIBO) stability of systems.

A system is BIBO **stable** when the output remains bounded for all time for every bounded input.

We can study the stability of a system either through its impulse response $h(t)$ or through its transfer function $H(s)$. An LTI system is BIBO stable if and only if its impulse response $h(t)$ is absolutely integrable; i.e.,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{3.98}$$

This condition is satisfied if the impulse response decays exponentially with increase in time.

We can determine whether a system is BIBO stable if its transfer function meets some conditions. The transfer function can be written

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \tag{3.99}$$

$H(s)$ must meet two requirements for the system to be stable. First, the numerator degree of $H(s)$ must not be larger than the denominator degree; otherwise, long division would produce

$$H(s) = k_n s^n + k_{n-1}s^{n-1} + \cdots + k_1 s + k_0 + \frac{R(s)}{D(s)} \tag{3.100}$$

where the degree of $R(s)$, the remainder of the long division, is less than the degree of $D(s)$. The inverse of $H(s)$ in eq. (3.100) does not meet the condition in eq. (3.98). Second, all the poles of the transfer function must be in the left half of the s-plane. That means that $H(s)$ must not have poles in the right half of the s-plane. Each pole must satisfy $p = -\alpha \pm j\omega$, where $\alpha > 0$; that is, the real part of the pole must be negative.

A system is BIBO **stable** if and only if all the poles of its transfer function $H(s)$ lie in the left half of the s-plane.

The system is considered unstable if one or more of its poles appear in the right half of the s-plane. If the transfer function has poles on the imaginary axis (i.e., with zero real parts), the system is said to be marginally stable.

Example 3.21

Check the stability of a system with transfer function

$$H(s) = \frac{s^2 + s - 10}{s^3 + 3s^2 + 4s + 2}$$

SOLUTION

$H(s)$ can be expressed in terms of its numerator $N(s)$ and denominator $D(s)$.

$$H(s) = \frac{N(s)}{D(s)} = \frac{s^2 + s - 10}{s^3 + 3s^2 + 4s + 2}$$

To determine the poles, we are only interested the roots of $D(s)$.

$$D(s) = s^3 + 3s^2 + 4s + 2 = (s+1)(s+1-j)(s+1+j)$$

The roots of $D(s) = 0$ or the poles of $H(s)$ are

$$p_1 = -1, \quad p_2 = -1 + j, \quad p_3 = -1 - j$$

All the three poles lie in the left half of the s-plane and we conclude that the system is stable.

Practice Problem 3.21 A system has its transfer function as

$$H(s) = \frac{s+6}{s^3 + s^2 + 2s + 8}$$

Check whether it is stable.

Answer: Unstable; has two poles in the right half of the s plane.

3.7 COMPUTING WITH MATLAB®

MATLAB is a powerful tool that can be used to do most of what we cover in this chapter. The impulse, step, ramp, sinusoidal, and exponential responses of continuous-time systems can be examined using MATLAB. In this section, we will examine how MATLAB can be used to do the following:

- Find the Laplace transform: MATLAB has a Laplace transform command X = **laplace**(x). If the expression produced by the command is complicated, we can use a MATLAB command **simplify**(x) to reduce the complexity.
- Find the inverse Laplace transform: This can be done in two ways. We may use the command **ilaplace** to invert $X(s)$. We may also use the command **residue**, which will produce a vector consisting of the residues of $X(s)$. We then apply Table 3.2 to find $x(t)$.
- Find the roots of a polynomial: By using the command **roots**, MATLAB can be used to find the roots of a polynomial or the zeros and poles of a transfer function.
- Find the time response: MATLAB can be used to find the impulse response of a system by using the command **impulse**. We can also find the step response of the system using the command **step**. To find the ramp response or any other time response, we can use the command **lsim**.
- Find the frequency response: The command **bode** produces Bode plots (both magnitude and phase) of a given transfer function $H(s)$. The frequency range and the number of points are automatically selected.

We will illustrate these with examples. There are many other commands (such as **conv, deconv, factor, pretty, lsim**, **freqs**, etc.) which are related to what we cover in this chapter; they can be examined by typing *help control*.

Example 3.22

Use MATLAB to find the Laplace transform of

$$x(t) = 2\delta(t) + e^{-3t}$$

SOLUTION

We recall that the commands **dirac**(*t*) and **heaviside**(*t*) are used to represent the unit impulse $\delta(t)$ and unit step $u(t)$, respectively. The MATLAB commands are:

```
syms x t
x = 2*dirac(t) + exp(-3*t);
X = laplace(x)
```

This produces the following result:

```
X =
1/(s+3) + 2
```

From this, we obtain

$$X(s) = 2 + \frac{1}{s+3}$$

Practice Problem 3.22 Use MATLAB to obtain the Laplace transform of

$$x(t) = u(t-1) - 2e^{-t}.$$

Answer: $X(s) = \frac{1}{s}e^{-s} - \frac{2}{s+1}$.

Example 3.23

Use MATLAB to find the inverse Laplace transform of

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

SOLUTION

The MATLAB code is as follows:

```
syms s v V
V =(10*s^2 +4)/( s*(s+1)*(s+2)^2 );
v = ilaplace(V)
```

This produces the following result:

```
v =
1-14*exp(-t)+(13+22*t)*exp(-2*t)
```

so that

$$v(t) = 1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t}, \quad t > 0$$

Practice Problem 3.23 Use MATLAB to obtain $g(t)$ if

$$G(s) = \frac{s^3 + 2s + 6}{s(s+1)^2(s+3)}$$

Answer: $(2 - 3.25e^{-t} - 1.5te^{-t} + 2.25\ e^{-3t})u(t)$.

Example 3.24

Use the **residue** command to find the Laplace inverse of

$$X(s) = \frac{4s^5 + 20s^4 + 16s^3 + 10s^2 - 12}{s^4 + 5s^3 + 8s^2 + 4s}$$

SOLUTION

This is an indirect way of finding the inverse Laplace transform. We specify the numerator (num) and the denominator (den) of the transfer function. We find the residues of $X(s)$ using the following code:

```
Num = [4 20 16 10 0 -12]; % numerator coefficients in
descending powers of s
den = [1 5 8 4 0]; % denominator coefficients in descending
powers of s
[r,p,k] = residue(num,den); % call residue
r =
 -15.0000
  46.0000
   2.0000
  -3.0000
p =
  -2.0000
  -2.0000
  -1.0000
        0
k =
   4    0
```

This produces a vector r that has the residues and a vector p that has the responding poles.

Notice that pole –2 is repeated. Also, since the order of the numerator of $X(s)$ is one greater than the order of the denominator of $X(s)$, k contains two values. From r, p, and k, we can write $X(s)$ as

$$X(s) = 4s + 0s^0 + \frac{-15}{s-(-2)} + \frac{46}{[s-(-2)]^2} + \frac{2}{s-(-1)} + \frac{-3}{s-0}$$
$$= 4s - \frac{15}{s+2} + \frac{46}{(s+2)^2} + \frac{2}{s+1} - \frac{3}{s}$$

Using Table 3.2, we obtain the inverse $v(t)$ as

$$x(t) = 4\delta'(t) - 15e^{-2t} + 46te^{-2t} + 2e^{-t} - 3u(t), \quad t \geq 0$$

Practice Problem 3.24 Given that $X(s) = \dfrac{5s^2}{(s+1)(s+3)}$, use the **residue** command to find $x(t)$.

Answer: $5\delta(t) - 22.5e^{-3t} + 2.5e^{-t}$, $t \geq 0$.

Example 3.25

Use MATLAB to find the zeros and poles of

$$H(s) = \frac{s^2 + 3s + 1}{s^3 + 4s^2 + 3s}$$

SOLUTION

We use the command **roots** to find the roots of the numerator to get the zeros and denominator to get the poles.

```
Num = [1 3 1];
den = [1 4 3 0];
z=roots(num);
p=roots(den);
```

This result is:

```
z =
  -2.6180
  -0.3820
p =
   0
  -3
  -1
```

Practice Problem 3.25 Repeat Example 3.25 for $H(s) = \dfrac{s^2 + 4s - 2}{s^3 + 4s^2 + 13s}$.

Answer: zeros: –4.4495, 0.4495, poles: 0, –2 + 3j, –2 –3j.

Example 3.26

Use MATLAB to plot the step response of a system whose transfer function is $H(s) = \dfrac{s+1}{(s+2)(s+3)}$

SOLUTION

We first need to expand the denominator.

$$H(s) = \frac{s+1}{s^2+5s+6}$$

By definition, the step response is the response when the input to the system is the unit step $u(t)$. Using the MATLAB script that follows, we obtain the response as plotted in Figure 3.19.

```
num = [1 1];
den = [1 5 6];
t = 0: 0.1: 4;
y=step(num,den,t);
plot(t,y)
```

Practice Problem 3.26 Use MATLAB to plot the impulse response of a system whose transfer function is given in Example 3.26. Hint: Use impulse(num,den,t).

Answer: See Figure 3.20.

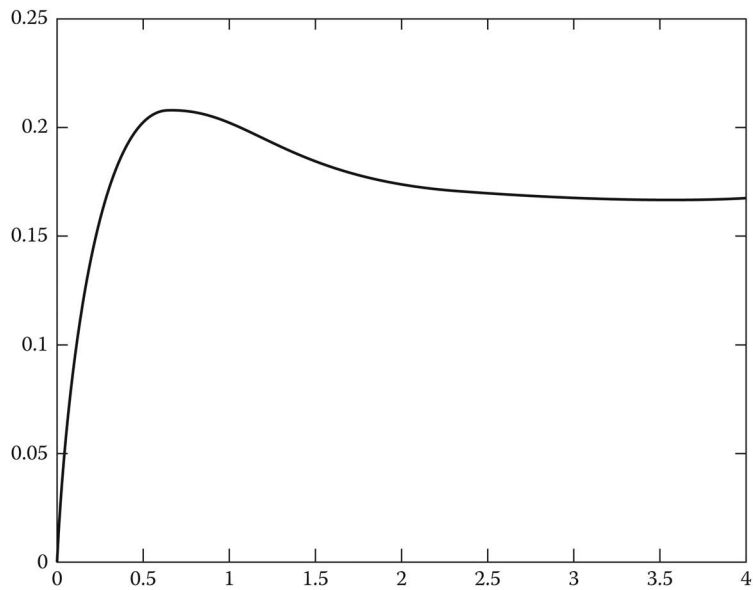


FIGURE 3.19 For Example 3.26.

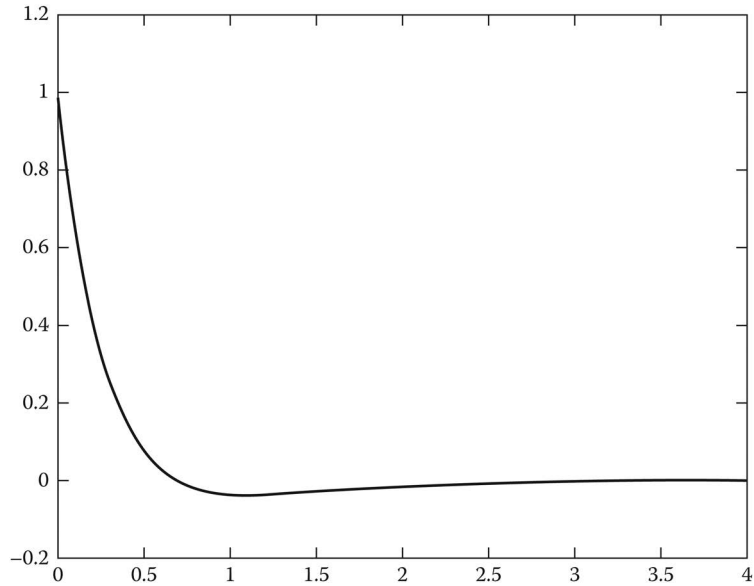


FIGURE 3.20 For Practice Problem 3.26.

Example 3.27

Use MATLAB to obtain the Bode plots for the transfer function

$$H(s) = \frac{100s}{s^2 + 12s + 20}$$

SOLUTION

The MATLAB script for the transfer is shown next, while the Bode plots are in Figure 3.21.

```
num = [100 0];
den = [1 12 20];
bode(num,den); %determines and draws Bode plots
```

Practice Problem 3.27 Repeat Example 3.27 for the transfer function

$$H(s) = \frac{s + 4}{s^2 + 5s + 8}$$

Answer: See Figure 3.22.

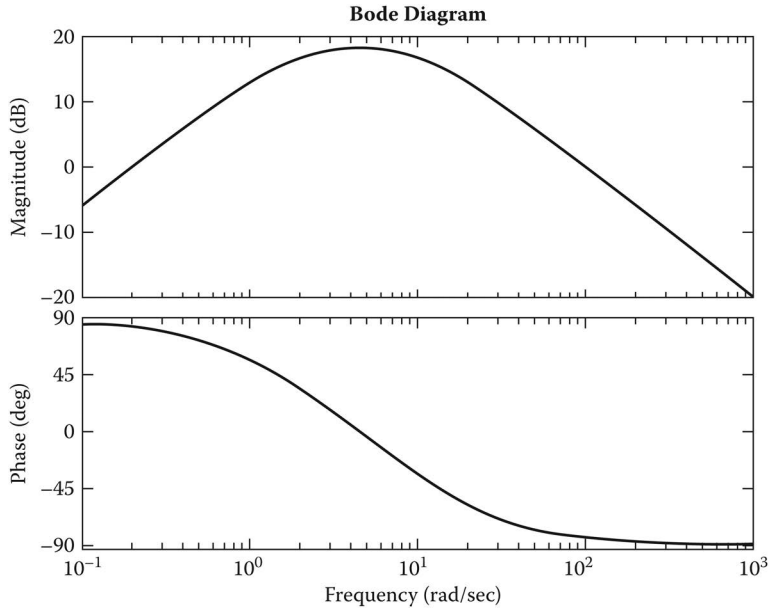


FIGURE 3.21 Magnitude and phase Bode plots for Example 3.27.

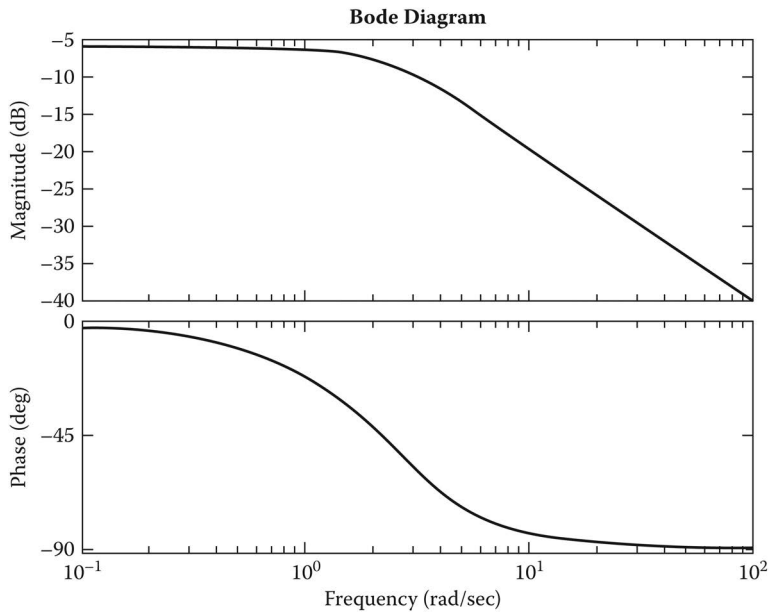


FIGURE 3.22 For Practice Problem 3.27.

SUMMARY

1. The Laplace transform converts a signal in the time domain to the frequency domain. Its unilateral form (used throughout this chapter) is defined as

$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

   The values of s for which $X(s)$ exists form the region of convergence.
2. Laplace transform properties are provided in Table 3.1, while the Laplace transforms of basic common functions are listed in Table 3.2.
3. The inverse Laplace transform, $\mathcal{L}^{-1}[X(s)]$, can be found by using partial fraction expansions along with the Laplace transform pairs in Table 3.2. The partial fraction expansion of $X(s)$ can be obtained when the poles of $X(s)$ are simple, repeated, complex conjugates or a combination of these.
4. The transfer function $H(s)$ of a system is the Laplace transform of the impulse response $h(t)$. $H(s)$ relates the output response $Y(s)$ and an input excitation $X(s)$; i.e., $H(s) = Y(s)/X(s)$, assuming all initial conditions are zero.
5. There are several applications of the Laplace transform. Some include the solution of integrodifferential equations, electric circuit analysis, and analysis of control systems.
6. Using the Laplace transform to analyze a circuit involves converting each element from the time domain to s-domain, solving the problem using circuit techniques, and converting the result to the time domain using the inverse Laplace transform.
7. A system is considered stable if all the poles of the transfer function $H(s)$ lie in the left half of the s-plane. Values of s for which $H(s) = 0$ and $H(s) = \infty$ are, respectively, zeros and poles of $H(s)$.
8. MATLAB is a convenient tool for finding the Laplace transform of a signal or its inverse. It is also used in obtaining the time response and frequency response of systems.

REVIEW QUESTIONS

3.1 A Laplace transform exists only in its region of convergence.
(a) True, (b) False

3.2 If $X(s)$ is the Laplace transform of $x(t)$, the variable s is known as
(a) Laplacian, (b) complex frequency, (c) transfer function, (d) zero, (e) pole.

3.3 The one-sided Laplace transform depends only on the value of the signal for $t \geq 0$.
(a) True, (b) False

3.4 The Laplace transform of $t^2u(t)$ is:
(a) $\frac{2}{s}$ (b) $\frac{2}{s^2}$ (c) $\frac{2}{s^3}$ (d) $\frac{e^{-2s}}{s}$

3.5 Let $X(s) = \dfrac{s+4}{(s+1)(s+2)(s+3)}$

The zero of $X(s)$ is at

(a) –4, (b) –3, (c) –2, (d) –1

3.6 For the function given in the previous question, the poles of $X(s)$ are at

(a) –4, (b) –3, (c) –2, (d) –1

3.7 Given

$$X(s) = \frac{s+3}{(s+1)(s+2)}$$

the initial value of $x(t)$ with transform is:

(a) nonexistent, (b) ∞, (c) 0, (d) 1, (e) 1/6

3.8 Let $X(s) = \dfrac{e^{-2s}}{s+4}$. The inverse Laplace transform $x(t)$ is

(a) $e^{-2(t-1)}u(t-1)$, (b) $e^{-4(t-2)}u(t-2)$. (c) $e^{-4t}u(t-2)$, (d) $e^{-(t-2)}u(t-4)$.

3.9 A network has the following transfer function:

$$H(s) = \frac{s+1}{(s+2)(s-3)}$$

Is the network stable?

(a) Yes, (b) No

3.10 The MATLAB® command for finding zeros and poles of a transfer function is:

(a) Laplace, (b) ilaplace, (c) roots, (d) step, (e) impulse

Answer: 3.1a, 3.2b, 3.3a, 3.4c, 3.5a, 3.6b,c,d, 3.7d, 3.8b, 3.9b, 3.10c.

PROBLEMS

Sections 3.2 and 3.3: Definition and Properties of the Laplace Transform

3.1 Obtain the Laplace transform of:

$$x(t) = \begin{cases} 2, & 0 < t < 2 \\ -1, & 2 < t < 3 \\ 0, & \text{otherwise} \end{cases}$$

3.2 Find the Laplace transform of

$$y(t) = \begin{cases} 5, & 0 < t < 2 \\ -5, & 2 < t < 4 \\ 10, & 4 < t < 6 \\ 0, & \text{otherwise} \end{cases}$$

3.3 Find $F(s)$ given that

$$f(t)=\begin{cases}2t, & 0<t<1\\ 2, & 1<t<2\\ 0, & \text{otherwise}\end{cases}$$

3.4 Find the Laplace transform of the following signals:

(a) $h(t)=tu(t-5)$

(b) $g(t)=4e^{-t}+e^{-3t}\sin 2t$

3.5 Find the Laplace transform of these signals:

(a) $x_1(t)=te^{2t}u(t)$

(b) $x_2(t)=t\cos 4t$

(c) $x_3(t)=\delta(t+2)$

(d) $x_4(t)=5\cos(2t+\pi/4)$

(e) $x_5(t)=t^2u(t-1)$

3.6 Show that:

(a) $\mathcal{L}\left[\cos^2(\omega t)u(t)\right]=\dfrac{s^2+2\omega^2}{s(s^2+4\omega^2)}$

(b) $\mathcal{L}\left[\sin^2(\omega t)u(t)\right]=\dfrac{2\omega^2}{s(s^2+4\omega^2)}$

3.7 Find the Laplace transform of $x(t)$ shown in Figure 3.23.

3.8 Obtain the Laplace transform of the signal shown in Figure 3.24.

3.9 Determine the Laplace transform of the signals $x(t)$ and $y(t)$ in Figure 3.25.

3.10 Using the Laplace transform, find the convolution $x(t)*y(t)$ for the following cases:

(a) $x(t)=e-2tu(t)$, $y(t)=tu(t)$

(b) $x(t)=e-tu(t)$, $y(t)=e-tu(t)$

(c) $x(t)=(\sin t)u(t)$, $y(t)=(\cos t)u(t)$

3.11 Given that $Y(s)=\dfrac{s-16}{s(s+8)}$, find $y(\infty)$.

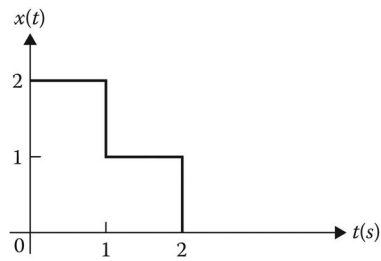


FIGURE 3.23 For Problem 3.7.

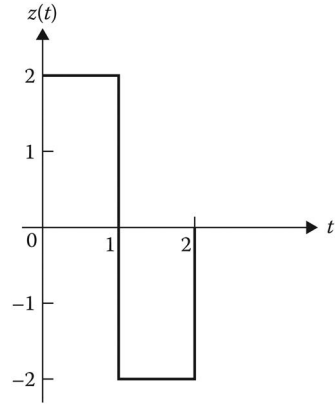


FIGURE 3.24 For Problem 3.8.

3.12 Using the initial value theorem, find $g(0)$ if

$$G(s) = \frac{s+1}{s^3+6s^2+11s+5}$$

3.13 Given that

$$F(s) = \frac{s^2+10s+6}{s(s+1)^2(s+2)}$$

Evaluate $f(0)$ and $f(\infty)$ if they exist.

3.14 Calculate the initial value of the signal $x(t) = 4u(t) + 2\cos t\, u(t)$.

3.15 Without determining $x(t)$, find the initial value of

$$X(s) = \frac{1}{s(s+2)}$$

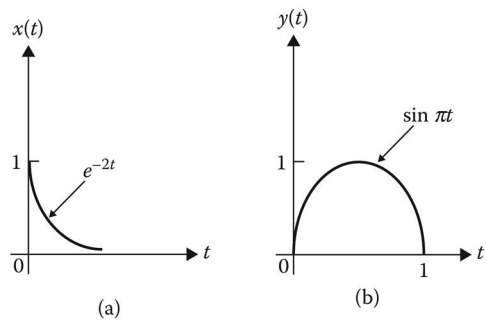


FIGURE 3.25 For Practice Problem 3.9.

3.16 Given that

$$X(s) = \frac{10}{s(s+1)(s+2)}$$

find $x(0)$ and $x(\infty)$ without determining $x(t)$.

3.17 Obtain the final value of $x(t)$ whose Laplace transform is

$$X(s) = \frac{20}{s(s+2)}$$

Section 3.4: The Inverse Laplace Transform

3.18 Obtain the inverse Laplace transform of the following functions:

(a) $X(s) = \frac{1}{s} + \frac{2}{s}e^{-s}$

(b) $Y(s) = \frac{10}{s^2 - 5s + 4}$

(c) $\frac{s-2}{s^2 + 2s + 10}$

3.19 Find the inverse Laplace transform of:

(a) $G(s) = \frac{s+2}{s^3 + 4s^2 + 3s}$

(b) $H(s) = \frac{10e^{-2s}}{(3s+5)^3}$

3.20 Find the inverse Laplace transform of the following functions:

(a) $F(s) = \frac{20(s+2)}{s(s^2 + 6s + 25)}$

(b) $P(s) = \frac{6s^2 + 36s + 20}{(s+1)(s+2)(s+3)}$

3.21 Find $y(t)$ that corresponds to

$$Y(s) = \frac{8e^{-s} - 4e^{-2s} - 2se^{-2s} - 6se^{-3s}}{s^2}$$

3.22 Determine the inverse Laplace transform of:

(a) $Z(s) = \frac{17}{s(s^2 + 6s + 34)}$

(b) $Y(s) = \frac{s+7}{s^2 + 8s + 20}$

(c) $X(s) = \frac{e^{-s}}{(s+1)(s+2)}$

3.23 Find the inverse Laplace transform of these functions:

(a) $X(s) = \dfrac{s^2 + 2s + 2}{s + 1}$

(b) $Y(s) = \dfrac{s^3 + 2s^2 + 3s + 1}{s^2(s + 1)}$

3.24 Find the inverse Laplace transform of:

(a) $H(s) = \dfrac{s + 4}{s(s + 2)}$

(b) $G(s) = \dfrac{s^2 + 4s + 5}{(s + 3)(s^2 + 2s + 2)}$

(c) $F(s) = \dfrac{e^{-4s}}{s + 2}$

(d) $D(s) = \dfrac{10s}{(s^2 + 1)(s^2 + 4)}$

3.25 Let $F(s) = \dfrac{5(s + 1)}{(s + 2)(s + 3)}$

(a) Use the initial and final value theorems to find $f(0)$ and $f(\infty)$.

(b) Verify your answer in part (a) by finding $f(t)$ using partial fractions.

Section 3.5: Transfer Function

3.26 Consider a system for which

$$\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + x(t)$$

Assuming that the system is causal, find the impulse response $h(t)$.

3.27 Obtain the impulse response of the system represented by the following equation:

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = x(t)$$

Assume zero initial conditions.

3.28 A causal LTI system has the transfer function:

$$H(s) = \frac{s + 2}{s^2 + 2s + 5}$$

Find the response $y(t)$ due to the input $x(t) = 2e^{-3t}\,u(t)$.

3.29 In the system shown in Figure 3.26, let

$$H_1(s) = \frac{2}{s + 2}, \quad H_2(s) = \frac{5}{s + 3}$$

Find the step response, i.e., find $y(t)$ when $x(t) = u(t)$.

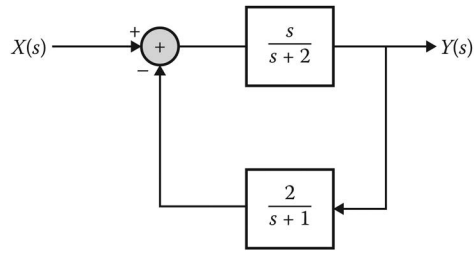


FIGURE 3.26 For Problem 3.29.

3.30 Find the transfer function of the system shown in Figure 3.27.
3.31 Find the transfer function of the feedback system shown in Figure 3.28.
3.32 Determine the transfer function of the system shown in Figure 3.29.
3.33 For the system shown in Figure 3.30, obtain the transfer function $H(s) = Y(s)/X(s)$.

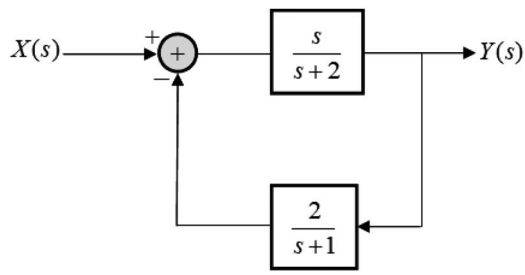


FIGURE 3.27 For Problem 3.30.

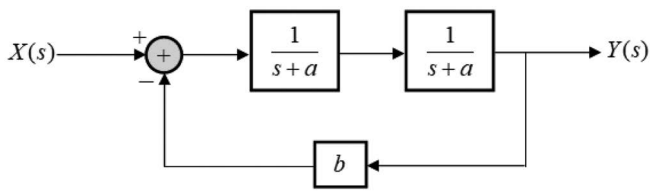


FIGURE 3.28 For Problem 3.31.

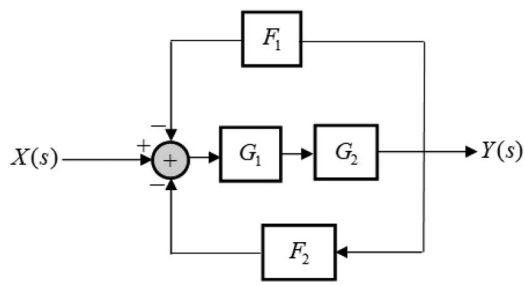


FIGURE 3.29 For Problem 3.32.

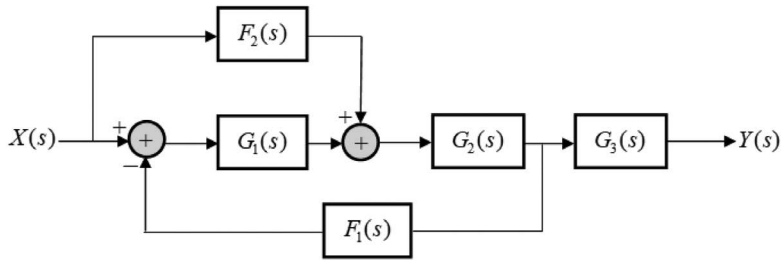


FIGURE 3.30 For Problem 3.33.

3.34 A manufacturing process is represented by the feedback control system shown in Figure 3.31. (a) Obtain the system transfer function. (b) Determine the system response $y(t)$ when $x(t) = u(t)$.

3.35 Determine the transfer function for the system shown in Figure 3.32.

3.36 Obtain the impulse response of a system with input $x(t)$ and output $y(t)$ as follows:

$$x(t) = 10u(t), \quad y(t) = \left(e^{-t} - 4e^{-2t}\right)u(t)$$

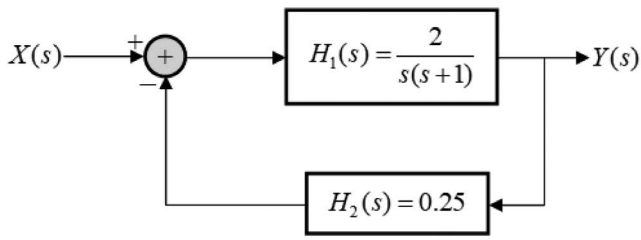


FIGURE 3.31 For Problem 3.34.

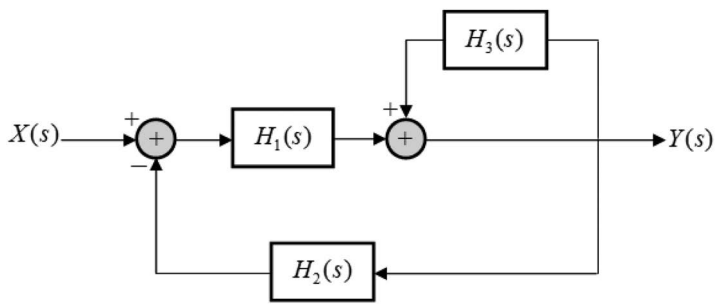


FIGURE 3.32 For Problem 3.35.

3.37 If the Laplace transfer function of a system is given by

$$H(s) = \frac{10}{(s+1)(s+2)}$$

Find the step response, i.e., the response when the input is a unit step.

3.38 A casual system has the transfer function

$$H(s) = \frac{20}{s+5}$$

Find the response $y(t)$ due to the input $x(t) = 2e^{-3t}u(t)$.

3.39 The transfer function of a system is given as

$$H(s) = \frac{s+5}{(s+2)(s+3)}$$

Find: (a) the impulse response
(b) the step response
(c) the ramp response due to $x(t) = r(t) = tu(t)$.

Section 3.6: Applications

3.40 Solve the differential equation

$$y''(t) + 7y'(t) + 12y(t) = e^{-t}u(t)$$

subject to $y(0) = -1$, $y'(0) = 2$.

3.41 Determine the complete response of the systems described by the following differential equation:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = \cos 2tu(t), \quad y(0^-) = 0, \quad y^{(1)}(0^-) = 1$$

3.42 Obtain the solution to this differential equation:

$$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = u(t), \quad y(0^-) = 1 = y^{(1)}(0^-)$$

3.43 A Butterworth high-pass filter with input $x(t)$ and output $y(t)$ is described by

$$\frac{d^2y(t)}{dt^2} + 1.41\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2}$$

Determine the impulse response.

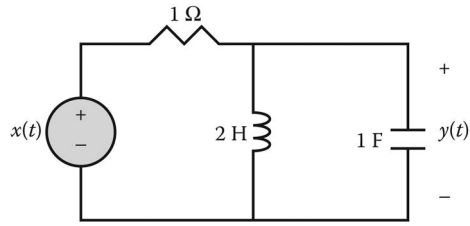


FIGURE 3.33 For Practice Problem 3.47.

3.44 Using the Laplace transform, solve the following differential equation for $t > 0$

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 5i = 2e^{-2t}$$

subject to $i(0) = 0$, $i'(0) = 2$.

3.45 An LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - 3x(t)$$

(a) Determine the transfer function of the system.
(b) Obtain the impulse response of the system.

3.46 Given that $v(0) = 2$, $dv(0)/dt = 4$, solve $\dfrac{d^2v}{dt^2} + 5\dfrac{dv}{dt} + 6v = 10e^{-t}u(t)$

3.47 In the circuit in Figure 3.33, assume that

$$x(t) = 4e^{-2t}u(t)$$

Find $y(t)$.

3.48 For the circuit in Figure 3.34, find the transfer function.

$$H(s) = \frac{V_o(s)}{V_i(s)}$$

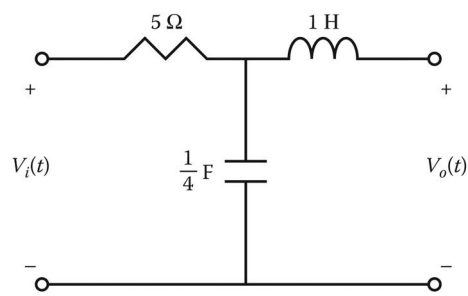


FIGURE 3.34 For Problem 3.48.

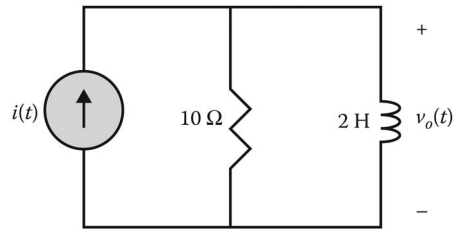


FIGURE 3.35 For Problem 3.49.

3.49 Consider the circuit in Figure 3.35. Find $vo(t)$ when:
(a) $i(t) = u(t)$ A
(b) $i(t) = 3e^{-t}u(t)$ A

3.50 Obtain in the transfer function of the circuit in Figure 3.36.

3.51 Consider the circuit shown in Figure 3.37. Find the transfer function $H(s)$.

3.52 Obtain the impedance $Z(s)$ of the network in Figure 3.38.

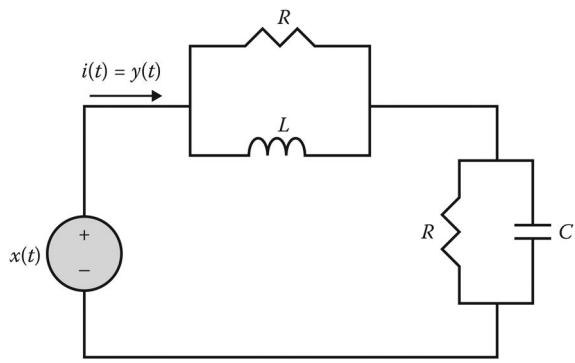


FIGURE 3.36 For Problem 3.50.

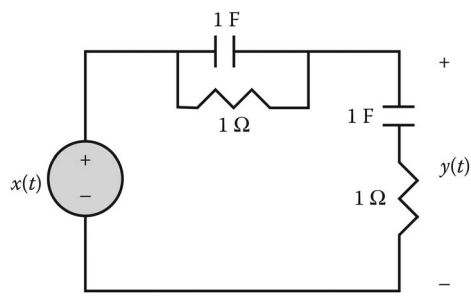


FIGURE 3.37 For Problem 3.51.

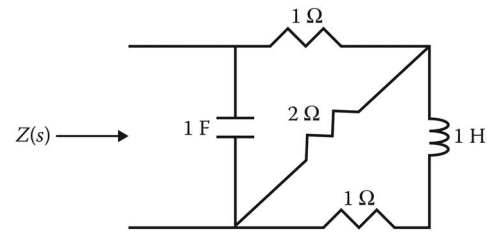


FIGURE 3.38 For Problem 3.52.

3.53 Use Laplace transformation to determine the mesh currents $I_1(s)$ and $I_2(s)$ in Figure 3.39. Let $v_s(t) = 2e^{-t}$ V.

3.54 Investigate the stability of these systems:

(a) $H(s) = \dfrac{s+2}{s^2-5s-6}$

(b) $H(s) = \dfrac{s+2}{s^2+9}$

(c) $H(s) = \dfrac{s^2+6}{s^2+5s+4}$

(d) $H(s) = \dfrac{s^3+10}{s^2+5s+6}$

3.55 Determine the stability of the systems represented by the following transfer functions:

(a) $H(s) = \dfrac{4}{s(s+1)}$

(b) $H(s) = -\dfrac{10s}{s^2+5s+6}$

(c) $H(s) = \dfrac{s^2+1}{s^2-6s+25}$

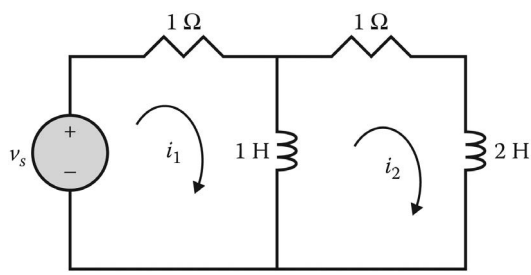


FIGURE 3.39 For Problem 3.53.

3.56 Given the impulse response $h(t)$ of an LTI system, determine if the system is BIBO stable.

(a) $h(t) = (t^3 + 2t^2 - 3t + 1)u(t)$

(b) $h(t) = e^{-t}\sin 4t, t \geq 0$

(c) $h(t) = \delta(t) + 2u(t)$

Section 3.7: Computing with MATLAB

3.57 Use MATLAB to find the Laplace transform of these functions for $t > 0$.

(a) $t^3 + t^2 + 2t - 4$

(b) $t\cos 3t$

(c) $e^{-t}\sin 2t$

3.58 Use MATLAB to find the inverse Laplace transform of the following functions:

(a) $\dfrac{1}{s} + \dfrac{1}{s}e^{-2s}$

(b) $\dfrac{2}{(s+1)^3}$

(c) $\dfrac{s^2 + 3s + 1}{s^2 + 4s + 8}$

3.59 Consider the function

$$H(s) = \frac{s^2 + 6s + 10}{s^3 + 7s^2 + 11s + 5}$$

Use the MATLAB **residue** function to obtain the inverse Laplace transform of $H(s)$.

3.60 Use MATLAB to find the inverse Laplace transform

$$X(s) = \frac{e^{-3s} + s^2 e^{-2s}}{s^3 + 3s^2 + 2s}$$

3.61 Use the MATLAB command **residue** to find the inverse Laplace transform of each of these functions:

(a) $X(s) = \dfrac{3s^2 + 6s - 10}{s^2 + 7s + 12}$

(b) $Y(s) = \dfrac{s^2 + 2s + 1}{s^3 + 4s^2 + 6s + 5}$

(c) $Z(s) = \dfrac{2s^3 + 12s^2 + 10s + 1}{s^5 + 16s^4 + 8s^3 + 2s}$

3.62 A linear system is represented by its transfer function

$$H(s) = \frac{2s+5}{(s+2)(s+3)}$$

Use MATLAB to determine and plot: (a) the impulse response, (b) the step response.

3.63 Use MATLAB to find the zeros and poles of these functions:

(a) $\dfrac{s-2}{(s+1)^2+9}$

(b) $\dfrac{s^2+2s+5}{s(s^2+4s+13)}$

(c) $\dfrac{s^2+10s+5}{s^3+4s^2+10s+6}$

3.64 Use MATLAB to determine if the systems represented by the following transfer functions are stable:

(a) $H(s) = \dfrac{s^3+12s^2+3s+10}{2s^4+20s^3+25s^2+15s+12}$

(b) $H(s) = \dfrac{s^4+2s^3-3s^2+s+1}{s^5+s^4+3s^3+3s^2+s+1}$

(c) $H(s) = \dfrac{4s^5+15s^4+25s^3+19s^2-14s-20}{s^6+8s^5+32s^4+64s^3+50s^2+24s+30}$

3.65 Obtain the Bode plots for the following transfer functions using MATLAB:

(a) $H(s) = \dfrac{s(s+10)}{(s+20)(s+50)}$

(b) $H(s) = \dfrac{s+1}{(s+2)(s^2+22.5s+16)}$

4 The Fourier Series

Work hard in silence, let your success be your noise.

Anonymous

HISTORICAL PROFILE

Jean Baptiste Joseph Fourier (1768–1830), French mathematician, known also as an Egyptologist and administrator, was the first to present the series and transform that bear his name. He was the first one to propose that the periodic waveforms could be represented by a sum of sinusoids. His major work, *The Analytic Theory of Heat* in 1822, stated the equations governing heat transfer in solids. Although Fourier's results were not enthusiastically received by the scientific world, his work provided the impetus for later work on trigonometric series and the theory of functions of a real variable.

DOI: 10.1201/9781003490869-4

Fourier was born in France in 1768, the son of a tailor. Joseph's mother died when he was nine years old, and his father died the following year. Fourier first attended the local military school conducted by Benedictine monks. He showed proficiency in mathematics and later became a teacher in mathematics at the same school. In 1793, he became involved in politics and joined the local Revolutionary Committee. Due to his political involvement, he narrowly escaped death twice. He died in his bed in Paris on May 16, 1830.

4.1 INTRODUCTION

This chapter is concerned with a means of analyzing systems with periodic excitations. The notion of periodic functions was introduced in Chapter 1, where the importance of sinusoids (or cosinusoids) was emphasized. In this chapter, we introduce Fourier series, a premier tool for analyzing periodic signals.

Although the Fourier series has a long history involving many individuals, the Fourier series is named after Jean Baptiste Joseph Fourier (1768–1830), a French mathematician and physicist. In 1822, Fourier was the first to suggest that any periodic function can be represented as a sum of sinusoids.

A **Fourier series** is a sum that represents a periodic function as an infinite sum of sines and cosines.

Fourier series allows us to model any periodic signal as a combination of simpler sine and cosine waves. It decomposes a time-domain signal into the frequencies that make it up. Such a representation provides a powerful tool for analyzing signals and systems. System responses to periodic signals are of practical interest because these signals are common.

Fourier series can only represent periodic functions. Nonperiodic functions are handled using the Fourier transform, which is an extension of Fourier series. Fourier analysis (Fourier series in this chapter and Fourier transform in the next chapter) plays a major role in system analysis for a number of reasons. First, Fourier analysis leads to the frequency spectrum of a continuous-time signal. The frequency spectrum displays the various sinusoidal components that make up the signal. Engineers think of signals in terms of their frequency spectra and of systems in terms of their frequency response. Second, Fourier analysis converts time-domain signals into frequency-domain representation, which lends new insight into the nature and the properties of the signals and systems. For many purposes, the frequency-domain representations are more convenient to analyze, synthesize, and process. Third, in the frequency domain, linear systems are described by linear algebraic equations, which can be easily solved, in contrast to the time-domain representation, where they are described by linear differential equations. It is for these reasons that Fourier analysis is used extensively today in science and engineering.

This chapter begins with the trigonometric Fourier series and how to determine the coefficients of the series. Then, we consider the complex exponential Fourier series. We will discuss some properties of Fourier series. We will cover circuit analysis, filtering, and spectrum analyzers as engineering applications of Fourier series. We will finally see how we can use MATLAB® to plot line spectra.

4.2 TRIGONOMETRIC FOURIER SERIES

The Fourier series can be represented in three ways: the sine-cosine, amplitude-phase, and complex exponential.

A periodic signal is one that repeats itself every T seconds. In other words, a continuous time signal $x(t)$ satisfies

$$x(t) = x(t + nT) \tag{4.1}$$

where n is an integer and T is the fundamental period of $x(t)$.

Any periodic function can be expressed as an infinite series consisting of sine or cosine functions. Thus, $x(t)$ can be expressed as

$$\begin{aligned} x(t) = a_0 + a_1 \cos\omega_0 t + b_1 \sin\omega_0 t + a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t \\ + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots \end{aligned} \tag{4.2}$$

This can be written as

$$x(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \tag{4.3}$$

where

$$\omega_o = \frac{2\pi}{T} \tag{4.4}$$

Here, ω_0 is known as the fundamental frequency (in rad/second). The coefficients a's are called the Fourier cosine coefficients, including a_0, the constant term, which is in reality the 0-th cosine term, and b's are called the Fourier sine coefficients. The Fourier coefficient a_0 in the previous equation is the constant or DC component of $x(t)$. The Fourier coefficients a_n and b_n (for $n \neq 0$) are the amplitudes of the sinusoids in the AC component of $x(t)$.

The **Fourier series** of a periodic signal $x(t)$ is a decomposition of $x(t)$ into a DC component and an AC component which consists of an infinite series of harmonic sinusoids.

A periodic function $x(t)$ can be expanded as a Fourier series only if it fulfills the Dirichlet conditions given as:

1. $x(t)$ should be integrable over any period; that is,

$$\int_{t_0}^{t_o+T} |x(t)|\,dt < \infty \tag{4.5}$$

2. $x(t)$ has only a finite number of maxima and minima over any period.
3. $x(t)$ has only a finite number of discontinuities over any period.

The process of determining the Fourier coefficients a_0, a_n, and b_n is called Fourier analysis. The coefficients can be determined as follows:

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt \tag{4.6}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt \tag{4.7}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt \tag{4.8}$$

The integration must be over a period T, regardless of where the integration begins. That is, the limits can be 0 to T, or $-t/2$ to $T/2$, or in general t_o to $T + t_o$. To evaluate the Fourier coefficient in eq. (4.5) requires the integrals in Sections A.5 and A.6 in Appendix A. Table 4.1 is useful in that it provides the values of the cosine, sine, and exponential functions for multiples of π (n = 1, 2, 3,…. is an integer).

Equation (4.3) is the sine-cosine form of Fourier series. An alternative representation is the amplitude-phase (or polar) form:

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\omega_0 t + \phi_n\right) \tag{4.9}$$

This combines the sine-cosine pair at frequency $n\omega_o$ into a single sinusoid. We can apply the trigonometric identity

$$\cos(A+B) = \cos A\ \cos B - \sin A\ \sin B \tag{4.10}$$

to the AC terms in eq. (4.13) to get

$$\begin{aligned} &a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \\ &= a_0 + \sum_{n=1}^{\infty} (A_n \cos\phi_n)\cos n\omega_0 t + \sum_{n=1}^{\infty} \left(-A_n \sin\phi_n\right)\sin n\omega_0 t \end{aligned} \tag{4.11}$$

TABLE 4.1
Evaluation of the Cosine, Sine, and Exponential Functions for Multiples of π

Function	Value
Cos $2n\pi$	1
Sin $2n\pi$	0
Cos $n\pi$	$(-1)^n$
Sin $2n\pi$	0
Cos $n\pi/2$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$
Sin $n\pi/2$	$\begin{cases} (-1)^{(n-1)/2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ j(-1)^{(n-1)/2}, & n = \text{odd} \end{cases}$

Source: C. K. Alexander and M.N.O. Sadiku, *Fundamentals of Electric Circuits.* New York: McGraw-Hill, 7th ed., 2021, p. 762.

When we equate the coefficients of the series expansions in eq. (4.3) and (4.11), we obtain

$$a_n = A_n \cos\phi_n, \quad b_n = -A_n \sin\phi_n \tag{4.12}$$

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = -\tan^{-1}\frac{b_n}{a_n} \tag{4.13}$$

These may also be related in a compact, complex form as

$$A_n \angle \phi_n = a_n - jb_n \tag{4.14}$$

The frequency components of the signal $x(t)$ can be displayed in terms of the amplitude and phase spectra. The frequency spectrum of $x(t)$ is the combination of both the amplitude and phase spectra.

Example 4.1

Given the periodic train of pulses shown in Figure 4.1, obtain Fourier coefficients a_o, a_n, and b_n. Plot the amplitude and phase spectra.

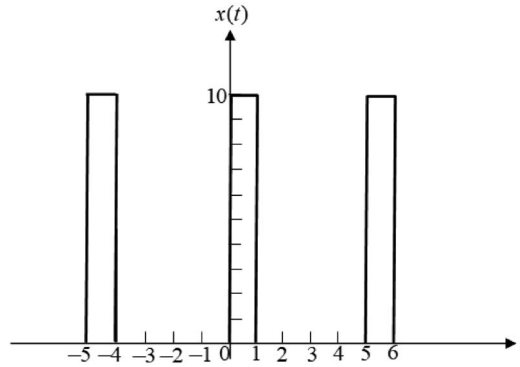


FIGURE 4.1 For Example 4.1.

SOLUTION

The signal can be expressed as

$$x(t) = \begin{cases} 10, & 0 < t < 1 \\ 0, & 1 < t < 5 \end{cases}$$

With $T = 5$, $\omega_o = 2\pi/5$ Using eq. (4.5a),

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{5}\left[\int_0^1 10\,dt + \int_1^5 0\,dt\right] = \frac{1}{5}10 = 2$$

Next, we use eq. (4.5b),

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_o t\,dt = \frac{2}{5}\int_0^1 10\cos n\omega_o t\,dt$$

$$= 4\frac{\sin n\omega_o t}{n\omega_o}\bigg|_0^1 = 4\frac{\sin n\omega_o}{n\omega_o}$$

Similarly, using eq. (4.5c) gives

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_o t\,dt = \frac{2}{5}\int_0^1 10\sin n\omega_o t\,dt$$

$$= -4\frac{\cos n\omega_o t}{n\omega_o}\bigg|_0^1 = 4\left[1 - \frac{\cos n\omega_o}{n\omega_o}\right]$$

The Fourier series expansion of the signal is

$$x(t) = 2 + 4\sum_{n=1}^{\infty}\left(\frac{\sin n\omega_o}{n\omega_o}\cos n\omega_o t + \left[1 - \frac{\cos n\omega_o}{n\omega_o}\right]\sin n\omega_o t\right)$$

TABLE 4.2
For Example 4.1

n	a_n	b_n	A_n	ϕ_n (Degrees)
1	3.0273	3.0164	4.2735	–44.89
2	0.9355	5.2876	5.3697	–79.97
3	–0.6237	4.8584	4.8983	82.68
4	–0.7568	3.7541	3.8296	78.60
5	0	3.3634	3.3634	90
6	0.5046	3.8361	3.8691	–82.51
7	0.2673	4.3679	4.3861	–86.50
8	–0.2339	4.3219	4.3282	86.90

Finally, we obtain the amplitude and phase spectra for the signal in Figure 4.1 by using

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

Applying these formulas for values of n = 1, 2,…, 8 produces the result in Table 4.2. From the table, the amplitude and phase spectra are plotted as shown in Figure 4.2.

Practice Problem 4.1 Obtain the Fourier series expansion for the waveform shown in Figure 4.3.

Answer: $y(t) = -0.5 + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{6}{n\pi} \sin n\pi t.$

Example 4.2

Find the Fourier series expansion of the backward sawtooth waveform of Figure 4.4.

SOLUTION

$$g(t) = 5 - 2.5t, \quad 0 < t < 2,$$

$$T = 2, \quad \omega_o = 2\pi/T = \pi$$

$$a_0 = \frac{1}{T}\int_0^T g(t)dt = \frac{1}{2}\int_0^2 (5 - 2.5t)dt = 0.5[5t - 2.5t^2/2]\Big|_0^2 = 2.5$$

$$a_n = \frac{2}{T}\int_0^T g(t)\cos n\omega_o t\, dt = \frac{2}{2}\int_0^2 (5 - 2.5t)\cos n\pi t\, dt$$

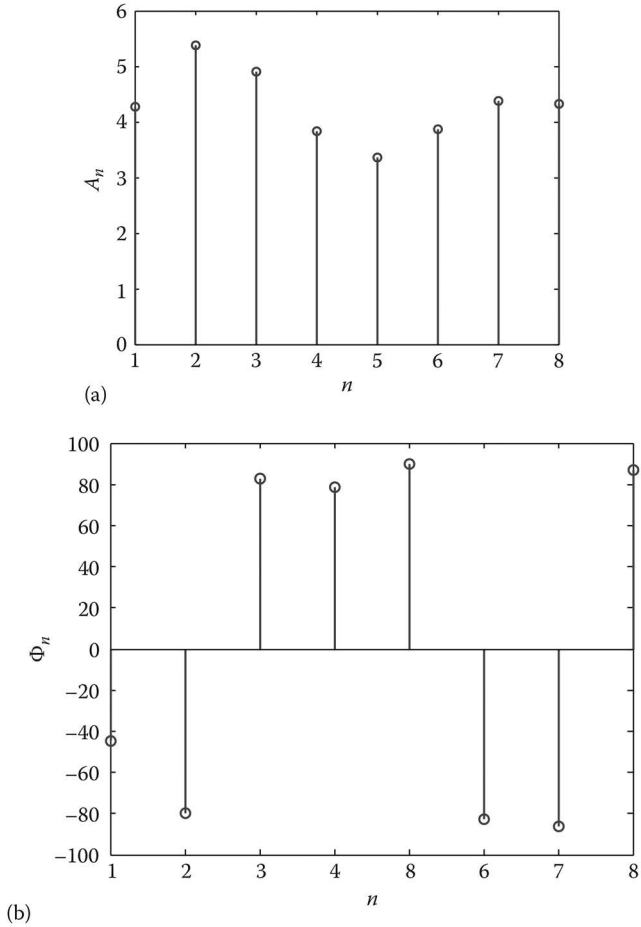


FIGURE 4.2 For Example 4.1: (a) amplitude spectrum and (b) phase spectrum.

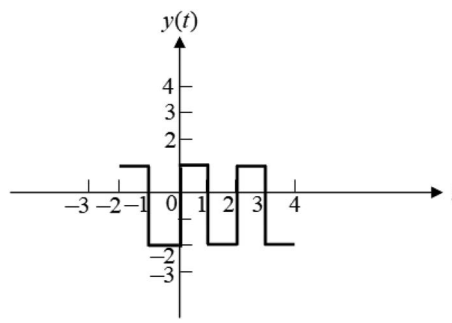


FIGURE 4.3 For Practice Problem 4.1.

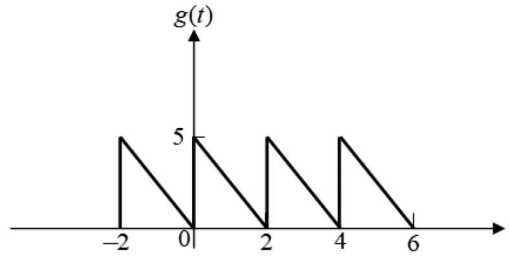


FIGURE 4.4 For Example 4.2.

Using the formulas from Appendix A.5,

$$\int \cos at = \frac{1}{a}\sin at$$

$$\int t\cos at = \frac{1}{a^2}\cos at + \frac{t}{a}\sin at$$

$$a_n = \frac{5}{n\pi}\sin n\pi t\Big|_0^2 + \left[\frac{2.5}{n^2\pi^2}\cos n\pi t + \frac{-2.5t}{n\pi}\sin n\pi t\right]_0^2$$

$$= 0 + \frac{2.5}{n^2\pi^2}(\cos 2n\pi - 1) - 0 = 0$$

Similarly,

$$b_n = \frac{2}{T}\int_0^T g(t)\sin n\omega_o t\, dt = \frac{2}{2}\int_0^2 (5 - 2.5t)\sin n\pi t\, dt$$

We use the formulas

$$\int \sin at = -\frac{1}{a}\cos at$$

$$\int t\sin at = \frac{1}{a^2}\sin at - \frac{t}{a}\cos at$$

$$b_n = -\frac{5}{n\pi}\cos n\pi t\Big|_0^2 + \left[\frac{-2.5}{n^2\pi^2}\sin n\pi t + \frac{2.5t}{n\pi}\cos n\pi t\right]_0^2$$

$$= 0 + 0 + \frac{5}{n\pi}(\cos 2n\pi - 0) = \frac{5}{n\pi}$$

Hence,

$$g(t) = 2.5 + \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t)$$

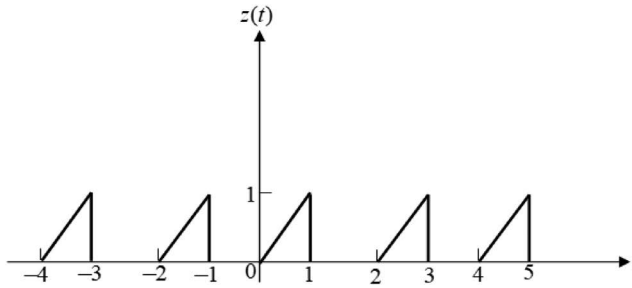


FIGURE 4.5 For Practice Problem 4.2.

Practice Problem 4.2 Obtain the trigonometric Fourier series for the signal in Figure 4.5.

Answer: $z(t) = 0.5 - \sum_{n=1}^{\infty} \left[\frac{\{1-(-1)^n\}}{n\pi} \cos n\pi t + \frac{(-1)^n}{n\pi} \sin n\pi t \right]$.

4.3 EXPONENTIAL FOURIER SERIES

The sine and cosine functions can be represented in terms of complex exponentials. It turns out that we can use complex exponentials to represent Fourier series. In many respects, this makes for a simpler representation.

The Fourier series representation of $x(t)$ can be expressed in complex exponential form as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{4.15}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency and the coefficients c_n are given by

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t}\, dt \tag{4.16}$$

The **exponential Fourier series** of a periodic signal $x(t)$ is a representation that is the sum of the complex exponentials at positive and negative harmonic frequencies.

By substituting the following Euler's identities in eq. (4.3)

$$\cos n\omega_0 t = \frac{1}{2}\left[e^{jn\omega_0 t} + e^{-jn\omega_0 t} \right] \tag{4.17a}$$

$$\sin n\omega_0 t = \frac{1}{2j}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right] = -\frac{j}{2}\left[e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right] \tag{4.17b}$$

we get

$$\begin{aligned} x(t) &= a_0 + \frac{1}{2}\sum_{n=1}^{\infty} a_n\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - j\, b_n\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) \\ &= a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(a_n - jb_n\right)e^{jn\omega_0 t} + \left(a_n + jb_n\right)e^{-jn\omega_0 t}\right] \end{aligned} \tag{4.18}$$

So the new coefficient c_n will be

$$\begin{aligned} c_0 &= a_0, \\ c_n &= \frac{\left(a_n - jb_n\right)}{2} = |c_n|\angle\theta_n, \\ c_{-n} &= c_n^* = \frac{\left(a_n + jb_n\right)}{2} \end{aligned} \tag{4.19}$$

so that $x(t)$ becomes

$$x(t) = c_0 + \sum_{n=1}^{\infty}\left[c_n e^{jn\omega_0 t} + c_{-n}\ e^{-jn\omega_0 t}\right] \tag{4.20a}$$

The concise form is

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} |c_n| e^{j(n\omega_0 t + \theta_n)} \tag{4.20b}$$

The complex Fourier coefficients c_n can be readily obtained as follows using eqs. (4.5b) and (4.5c) for a_n, b_n.

$$c_0 = a_0 = \frac{1}{T}\int_0^T x(t)\,dt \tag{4.21a}$$

For $n = 1, 2, 3,\ldots$we have

$$c_n = \frac{\left(a_n - jb_n\right)}{2} = \frac{1}{T}\int_0^T x(t)\,(\cos n\omega_0 t - j\sin n\omega_0 t)\,dt = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt \tag{4.21b}$$

Similarly

$$c_{-n} = c_n^* = \frac{(a_n + jb_n)}{2} = \frac{1}{T}\int_0^T x(t)e^{jn\omega_0 t}\,dt$$

The last expression is equivalent to stating that for $n = -1, -2, -3, \ldots\ldots$

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt \tag{4.21c}$$

The three eqs. (4.21a), (4.21b), and (4.21c) can be combined into one expression

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt \quad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{4.22}$$

Therefore, the complex Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{4.23}$$

Eq. (4.23) is also called the complex frequency spectrum of $x(t)$, which is formed by both the complex amplitude spectrum (even symmetry) and the phase spectrum (odd symmetry).

The three forms of Fourier series are now summarized as follows:

Sine-Cosine Form

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_o t + b_n \sin n\omega_o t\right) \tag{4.24}$$

Amplitude-Phase Form

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t + \phi_n) \tag{4.25}$$

Complex Exponential Form

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{4.26}$$

The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$A_n \angle \phi_n = a_n - jb_n = 2c_n \tag{4.27}$$

or

$$c_n = |c_n| \angle \theta_n = \frac{\sqrt{a_n^2 + b_n^2}}{2} \angle -\tan^{-1} b_n/a_n = A_n \angle \phi_n \tag{4.28}$$

if only $a_n > 0$. Note that the phase θ_n of c_n is equal to ϕ_n.

Example 4.3

Find the exponential Fourier series for $x(t) = t$, $-1 < t < 1$, with $f(t + 2n) = f(t)$ and n is an integer. Plot the amplitude and phase spectra.

SOLUTION

Since $T = 2$, $\omega_0 = 2\pi/T = \pi$ Hence

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_o t}dt = \frac{1}{2}\int_{-1}^{1} te^{-jn\pi t}\,dt$$

Using integration by parts,

$$u = t \rightarrow du = dt$$

$$dv = \frac{1}{2}e^{-jn\pi t}dt \rightarrow v = -\frac{1}{2jn\pi}e^{-jn\pi t}$$

$$c_n = -\frac{t}{2jn\pi}e^{-jn\pi t}\Big|_{-1}^{1} + \frac{1}{2jn\pi}\int_{-1}^{1} e^{-jn\pi t}\,dt$$

$$= \frac{j}{2n\pi}\left[e^{-jn\pi} + e^{jn\pi}\right] + \frac{1}{2n^2\pi^2(-j)^2}e^{-jn\pi t}\Big|_{-1}^{1}$$

$$= \frac{j}{n\pi}\cos n\pi - \frac{1}{2n^2\pi^2}(e^{-jn\pi} - e^{jn\pi})$$

$$c_n = \frac{j(-1)^n}{n\pi} + \frac{2j}{2n^2\pi^2}\sin(n\pi) = \frac{j(-1)^n}{n\pi}, \quad n \neq 0$$

We treat $n = 0$ as a special case.

$$c_0 = \frac{1}{2}\int_{-1}^{1} te^{-0}\,dt = \frac{1}{2}\frac{t^2}{2}\Big|_{-1}^{1} = \frac{1}{4}(1-1) = 0$$

TABLE 4.3
For Example 4.3

N	$\lvert c_n \rvert$	ϕ_n (Degrees)
–3	0.1061	90
–2	0.1592	–90
–1	0.3183	90
0	0	0
1	0.3183	–90
2	0.1592	90
3	0.1061	–90

Thus,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{j}{n\pi} e^{jn\pi t}$$

For different values of n, we calculate $|c_n|$ and θ_n as shown in Table 4.3. By plotting $|c_n|$ and θ_n for different n, we obtain the amplitude spectrum and the phase spectrum shown in Figure 4.6. Notice that the amplitude spectrum is an even function, while the phase spectrum is odd, as expected.

Practice Problem 4.3 For the periodic function $y(t) = t^2$, $0 < t < T$, obtain the complex Fourier series for the case $T = 2$. Determine the amplitude and phase spectra.

Answer: $y(t) = 1.333 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n^2\pi^2}(1 + jn\pi)e^{jn\pi t}$

See Figure 4.7 for the spectra.

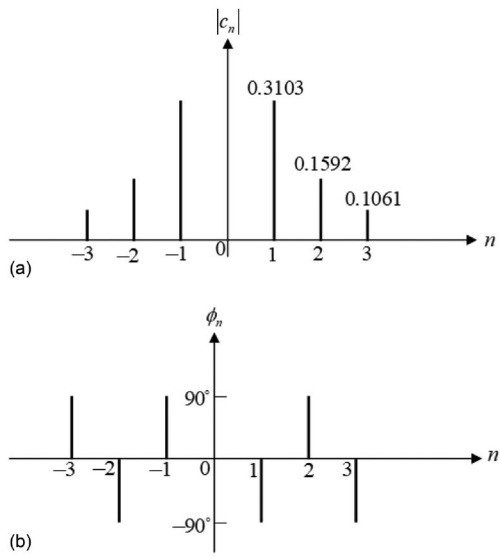


FIGURE 4.6 For Example 4.3.

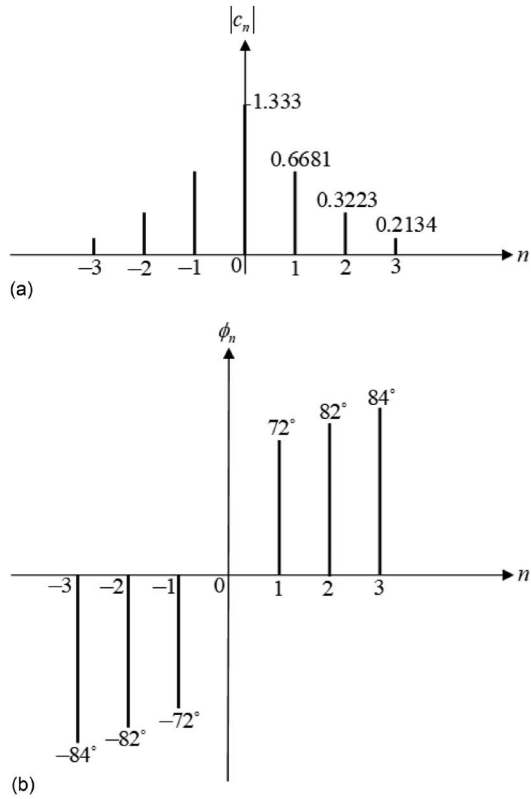


FIGURE 4.7 For Practice Problem 4.3.

Example 4.4

Find the complex Fourier series of the signal in Figure 4.8.

SOLUTION

$$T = 2,\ \omega_o = 2\pi/T = \pi$$

$$f(t) = \begin{cases} 5(1-t), & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

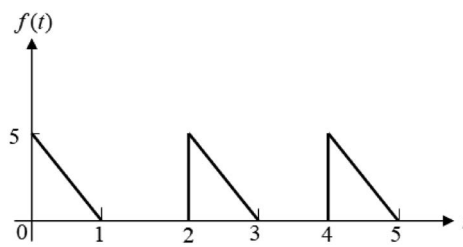


FIGURE 4.8 For Example 4.4.

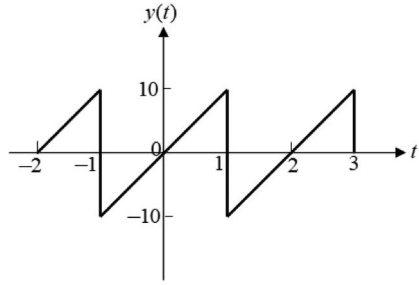


FIGURE 4.9 For Practice Problem 4.4.

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2}\int_0^1 5(1-t)e^{-jn\pi t}\,dt$$

$$= \frac{5}{2}\int_0^1 e^{-jn\pi t}\,dt - \frac{5}{2}\int_0^1 te^{-jn\pi t}\,dt = \frac{5}{2}\frac{e^{-jn\pi t}}{-jn\pi}\bigg|_0^1 - \frac{5}{2}\frac{e^{-jn\pi t}}{(-jn\pi)^2}(-jn\pi t - 1)\bigg|_0^1$$

$$= \frac{5}{2}\frac{\left[e^{-jn\pi}-1\right]}{-jn\pi} - \frac{5}{2}\frac{e^{-jn\pi}}{-n^2\pi^2}(-jn\pi - 1) + \frac{5}{2}\frac{(-1)}{-n^2\pi^2}$$

But $e^{-jn\pi} = \cos\pi n - j\sin n\pi = \cos n\pi + 0 = (-1)^n$

$$c_n = \frac{2.5[1-(-1)^n]}{jn\pi} - \frac{2.5(-1)^n[1+jn\pi]}{n^2\pi^2} + \frac{2.5}{n^2\pi^2}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi t}$$

Practice Problem 4.4 Obtain the exponential Fourier series expansion of $y(t)$ shown in Figure 4.9.

Answer: $y(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{j10}{n\pi} e^{jn\pi t}$.

4.4 PROPERTIES OF FOURIER SERIES

Some properties of Fourier series are discussed in this section. These properties provide us a better understanding of the Fourier series.

4.4.1 Linearity

The Fourier series expansion for periodic signals $x(t)$ and $y(t)$ with the same period is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \propto_n \exp\left[jn\omega_0 t\right] \tag{4.29}$$

$$y(t) = \sum_{n=-\infty}^{\infty} \beta_n \exp\left[jn\omega_0 t\right] \tag{4.30}$$

If $z(t)$ is a linear combination of $x(t)$ and $y(t)$, then

$$z(t) = k_1 x(t) + k_2 y(t) \tag{4.31}$$

where k_1 and k_2 are arbitrary constants. Then we can write,

$$z(t) = \sum_{n=-\infty}^{\infty} (k_1 \propto_n + k_2 \beta_n) \exp\left[jn\omega_0 t\right] = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left[jn\omega_0 t\right] \tag{4.32}$$

This implies that the Fourier coefficients are

$$\gamma_n = k_1 \propto_n + k_2 \beta_n \tag{4.33}$$

The linearity property is easily extended to a linear combination of an arbitrary number of signals with period T.

4.4.2 Time Shifting

When a time shift is applied to a periodic signal $x(t)$, the period T of the signal is preserved. The Fourier series coefficient γ_n of the resulting signal $x(t - \tau)$ may be expressed as

$$\begin{aligned} \gamma_n &= \frac{1}{T}\int_0^T x(t-\tau)\exp\left[-jn\omega_0 t\right]dt = \exp\left[-j\ \omega_0\tau\right]\frac{1}{T}\int_0^T x(\lambda)\exp\left[-jn\omega_0\lambda\right]d\lambda \\ &= \propto_n \exp\left[-jn\omega_0\tau\right] \end{aligned} \tag{4.34}$$

where $\propto_n$ is the nth Fourier series coefficient of $x(t)$. Here we have changed variables by letting $\lambda = t - \tau$. That is, if

$$x(t) \overset{FS}{\longrightarrow} \alpha_n$$

then

$$x(t-\tau) \overset{FS}{\longrightarrow} \alpha_n e^{-jn\omega_0\tau} \tag{4.35}$$

One consequence of this property is that when a periodic signal is shifted in time, the magnitude of the Fourier coefficient remains unaltered, that is $|\gamma_n| = |\propto_n|$.

4.4.3 Time Reversal

The time-reversal property, when applied to a continuous-time signal, results in a time reversal of the corresponding sequence of Fourier series coefficients. To determine the Fourier series coefficients of $y(t) = x(-t)$, consider

$$x(-t) = \sum_{n=-\infty}^{\infty} \propto_n \exp\left[-\frac{jn\pi t}{T}\right] \tag{4.36}$$

By substituting $n = -m$, we get

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} \propto_{-m} \exp\left[\frac{jm\pi t}{T}\right] \tag{4.37}$$

Thus, we conclude that

$$\begin{aligned} x(t) &\overset{\mathcal{FS}}{\rightarrow} \alpha_n \\ x(-t) &\overset{\mathcal{FS}}{\rightarrow} \alpha_{-n} \end{aligned} \tag{4.38}$$

An interesting consequence of time reversal is that when $x(t)$ is even, that is, if $x(-t) = x(t)$, then its Fourier series coefficients are also even, that is $\propto_{-n} = \propto_n$. Similarly when $x(t)$ is odd, that is, if $x(-t) = -x(t)$, then its Fourier series coefficients are also odd, that is $\propto_{-n} = -\propto_n$.

4.4.4 Time Scaling

The time-scaling operation applies directly to each of the harmonic components of $x(t)$. If $x(t)$ has the Fourier series representation $x(t) = \sum_{n=-\infty}^{\infty} \propto_n \exp[jn\omega_0 t]$, then the Fourier series representation of the time-scaled signal $x(at)$ is

$$x(at) = \sum_{n=-\infty}^{\infty} \propto_n \exp[jn\omega_0 at] \tag{4.39}$$

that is,

$$x(at) \overset{\mathcal{FS}}{\rightarrow} \alpha_n \tag{4.40}$$

The Fourier series coefficients have not changed; the Fourier series representation has changed because of the change in the fundamental frequency $a\omega_0$.

4.4.5 Even and Odd Symmetries

Three types of symmetries need to be discussed: even, odd, and half-wave odd symmetry.

A function $x(t)$ is even if $x(t) = x(-t)$, and odd if $x(t) = -x(-t)$. Any function $x(t)$ is a sum of an even function and an odd function, and this can be done in only one way.

We know that any periodic function $x(t)$, with period 2π, has a Fourier expansion of the form

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \tag{4.41}$$

If $x(t)$ is even, then $b_n = 0$ and the Fourier series in eq. (4.3) becomes

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) \tag{4.42}$$

If $x(t)$ is odd, then $a_0 = 0 = a_n$ and the Fourier series is simply

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{4.43}$$

A periodic function possesses half-wave symmetry if it satisfies the constraints

$$x(t) = -x\left(t - \frac{T}{2}\right) \tag{4.44}$$

If a function is shifted one half-period and inverted, it looks identical to the original, and then it is called a half-wave symmetry. Examples of half-wave odd symmetric functions are shown in Figure 4.10. A half-wave odd symmetry can be even, odd, or neither.

$$a_0 = 0, \quad a_n = 0 \quad \text{and} \quad b_n = 0 \quad \text{for } n \text{ even}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos n\omega_0 t\, dt, \quad \text{for } n \text{ odd} \tag{4.45}$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin n\omega_0 t\, dt, \quad \text{for } n \text{ odd}$$

This shows that the Fourier series of a half-wave symmetric function contains only odd harmonics.

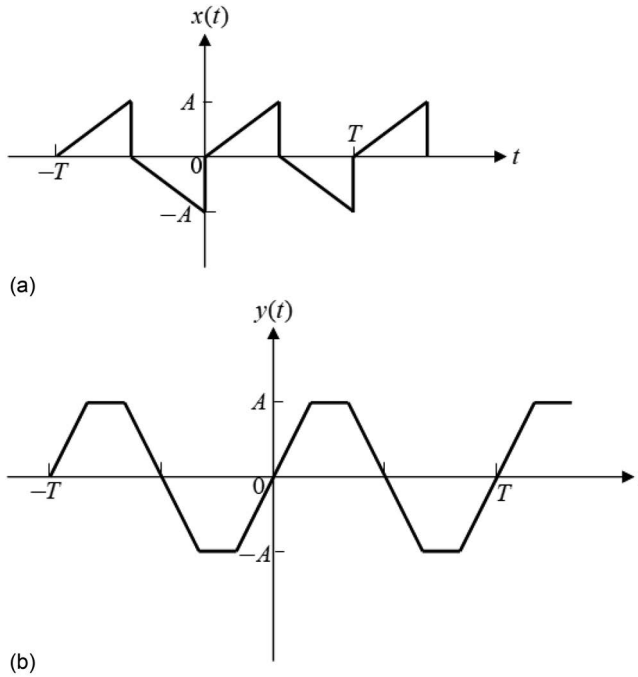


FIGURE 4.10 Typical examples of half-wave odd symmetric functions.

4.4.6 Parseval's Theorem

Parseval's theorem states that if $x(t)$ is a periodic function with period T, then the average power P of the signal is defined by

$$P = \frac{1}{T}\int_0^T x^2(t)\,dt \tag{4.46}$$

Again let $x(t)$ be an arbitrary periodic signal with period T, and consider the Fourier series of $x(t)$ given by eq. (4.15). By Parseval's theorem, the average power P of the signal $x(t)$ is given by

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{4.47}$$

Parseval's theorem states that the average power of the signal $x(t)$ over one period equals the sum of the squared magnitudes of all the complex Fourier coefficients.

To prove Parseval's theorem, assume $x(t)$ has a complex Fourier series of the usual form:

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}, \quad \omega_0 = \left(\frac{2\pi}{T}\right) \\
c_n &= \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t}\, dt \\
x^2(t) &= x(t)\,x(t) = x(t)\sum c_n\, e^{jn\omega_0 t} = \sum c_n\, x(t) e^{jn\omega_0 t}\, dt \\
P &= \frac{1}{T}\int_0^T x^2(t)\, dt = \frac{1}{T}\int_0^T \sum c_n\, x(t) e^{jn\omega_0 t}\, dt \\
&= \frac{1}{T}\sum c_n \int_0^T x(t) e^{jn\omega_0 t}\, dt \\
&= \sum c_n \cdot c_n^* \\
P &= \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}
\tag{4.48}
$$

Parseval's theorem can also be written in terms of the Fourier coefficients a_n, b_n of the trigonometric Fourier series. The engineering interpretation of this theorem is as follows. Suppose $x(t)$ denotes an electrical signal (current or voltage), then from elementary circuit theory $x^2(t)$ is the instantaneous power (in a 1-ohm resistor) so that

$$P = \frac{1}{T}\int_0^T x^2(t)\, dt \tag{4.49}$$

is the energy dissipated in the resistor during one period. In terms of the trigonometric Fourier series, eq. (4.48) becomes

$$P = a_o^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{4.50}$$

In terms of the amplitude-phase Fourier series, we have

$$P = a_o^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2 \tag{4.51}$$

Table 4.4 summarizes the power associated with Fourier series coefficients.

TABLE 4.4
Power Associated with Fourier Series Coefficients

Term	Sine-Cosine Form	Amplitude-Phase Form	Exponential Form
DC term	a_o^2	a_o^2	$\lvert c_0\rvert^2$
Others	$\frac{1}{2}\left(a_n^2 + b_n^2\right)$	$\frac{1}{2}A_n^2$	$2\lvert c_n\rvert^2 = \lvert c_n\rvert^2 + \lvert c_{-n}\rvert^2$

Example 4.5

The waveform in Figure 4.11(a) has the following Fourier series:

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\omega_0 t$$

Obtain the Fourier series of $y(t)$ in Figure 4.11(b).

SOLUTION

If we take the derivative of $x(t)$ in Figure 4.11(a), we obtain $y(t)$ in Figure 4.11(b).

$$\begin{aligned}\frac{dx}{dt} &= \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\omega_0\sin(2n-1)\omega_0 t, \quad \omega_0 = \frac{2\pi}{T} \\ &= \frac{8A}{\pi T}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega_0 t\end{aligned} \tag{4.5.1}$$

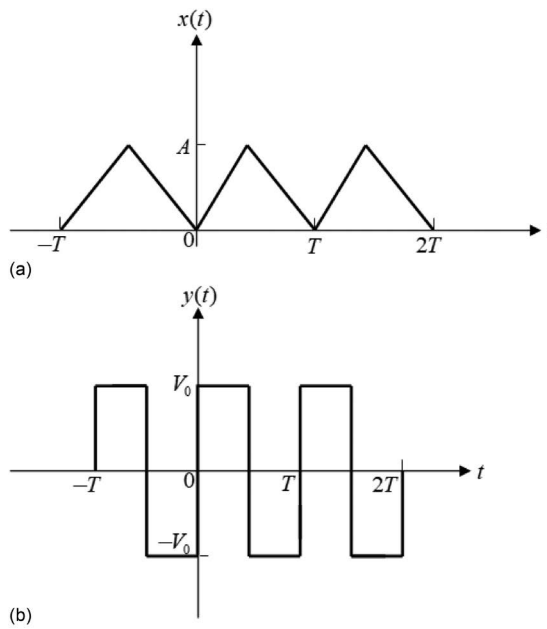


FIGURE 4.11 For Example 4.5.

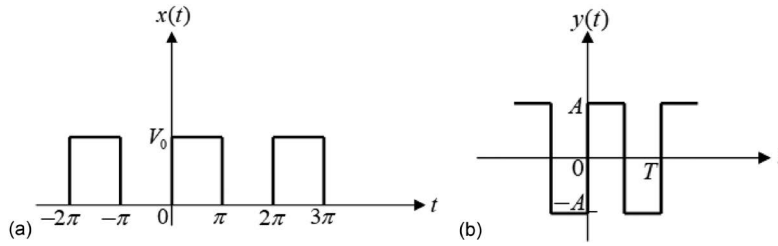


FIGURE 4.12 For Practice Problem 4.5.

But the slope of $x(t)$ is the amplitude of $y(t)$, i.e.,

$$\text{slope} = \frac{A}{T/2} = \frac{2A}{T} = V_o \rightarrow A = \frac{V_o T}{2} \tag{4.5.2}$$

Substituting for $x(t)$ into eq. (4.5.1) gives

$$y(t) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)\omega_0 t$$

Practice Problem 4.5 Obtain the Fourier series of the function $x(t)$ in Figure 4.12(a) from the Fourier series of $y(t)$ in Figure 4.12(b), which is given as

$$y(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)\omega_0 t$$

Answer: $x(t) = \dfrac{V_o}{2} + \dfrac{2V_o}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)t.$

Example 4.6

For the waveform shown in Figure 14.13:

a. Specify the type of symmetry it has.
b. Calculate a_3 and b_3.

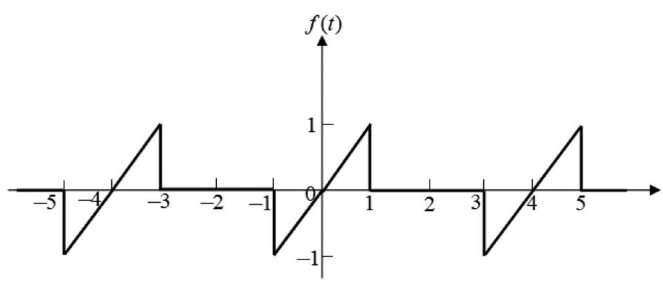


FIGURE 4.13 For Example 4.6.

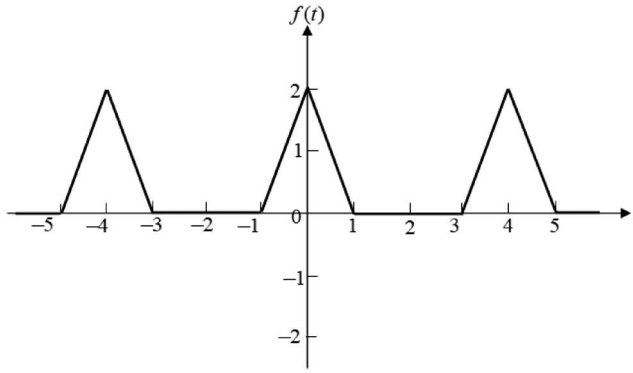


FIGURE 4.14 For Practice Problem 4.6.

SOLUTION

a. This is an odd function so that $a_0 = a_n = 0$.
b. $T = 4$, $\omega_o = 2\pi/T = \pi/2$

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$b_n = \frac{4}{4}\int_0^1 t \sin\frac{n\pi t}{2}\,dt = \left[\frac{4}{n^2\pi^2}\sin\frac{n\pi t}{2} - \frac{2t}{n\pi}\cos\frac{n\pi t}{2}\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\cos\frac{n\pi}{2} - 0$$

$$b_n = \begin{cases} \dfrac{4(-1)^{(n-1)/2}}{n^2\pi^2}, & n = \text{odd} \\ \dfrac{-2(-1)^{n/2}}{n\pi}, & n = \text{even} \end{cases}$$

$$a_3 = 0,\ b_3 = 4(-1)/(9\pi^2) = -0.04503$$

Practice Problem 4.6 Determine the trigonometric Fourier series expansion of the signal in Figure 4.14. What type of symmetry does it have?

Answer: $f(t) = \dfrac{1}{2} + \dfrac{8}{\pi^2}\sum_{k=1}^{\infty}\dfrac{1}{n^2}\left[1 - \cos\left(\dfrac{n\pi}{2}\right)\right]\cos\left(\dfrac{n\pi t}{2}\right)$, even.

Example 4.7

Calculate the average power in the signal

$$x(t) = 4 + 2\cos(2t - 30^\circ) + \cos(4t - 15^\circ).$$

SOLUTION

Using eq. (4.51), we get

$$P = 4^2 + \frac{1}{2}(2^2 + 1^2) = 18.5$$

Practice Example 4.7 Determine the average power in the signal

$$x(t) = 1 + 4\cos(60\pi t + 10°) + 2\cos(120\pi t - 30°)$$

Answer: 11.

4.5 TRUNCATED COMPLEX FOURIER SERIES

The Fourier series representation of $x(t)$ can be expressed in complex exponential form as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{4.52}$$

It is an infinite summation, and to truncate $x(t)$ to finite partial sum of $x_N(t)$ at $n = N$, we can write eq. (4.52) as follows:

$$x_N(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} \tag{4.53}$$

We expect the signal approximation $x_N(t)$ to converge to $x(t)$ as $N \to \infty$. The behavior of the partial sum in the vicinity of discontinuity exhibits ripples, and the peak amplitude of these ripples does not seem to decrease with increasing N. Gibbs showed that for the discontinuity of unit height, the partial sum exhibits an overshoot of 9% of the height of the discontinuity, no matter how large N becomes. Thus, as N increases, the ripples in the partial sum become compressed toward the discontinuity, but for any finite value of N, the peak amplitude of the ripples remains constant. This is known as Gibbs phenomenon. The Gibbs phenomenon (also known as ringing artifacts) is named after the American physicist J. Willard Gibbs.

Gibbs discovered that the truncated Fourier series approximation $x_N(t)$ of a signal $x(t)$ with a discontinuity will in general exhibit high-frequency ripples and overshoot $x(t)$ near the discontinuities. If such an approximation is used in practice, a large enough value of N should be chosen so as to guarantee that the total energy in this ripple is insignificant. These are shown in the Figure 4.15 for various values of N. We know that the energy in the approximation error vanishes and that the Fourier series representation of a discontinuous signal such as the square wave converges.

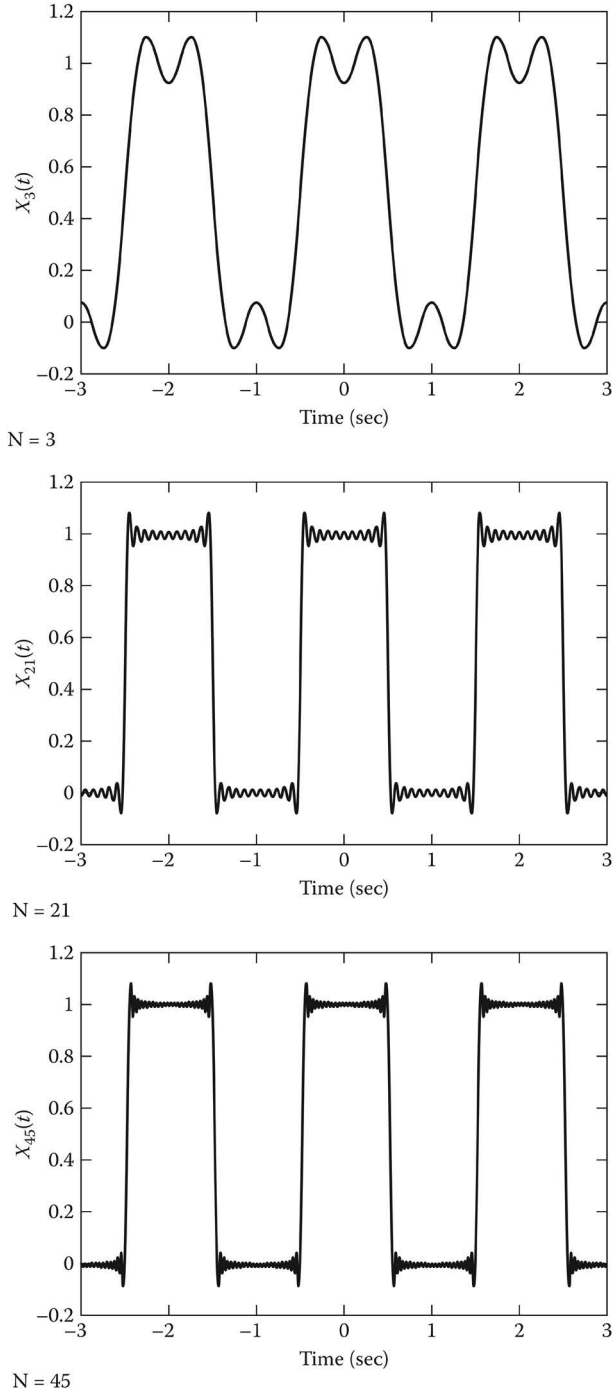


FIGURE 4.15 Convergence of the Fourier series representation of a square wave for several values of *N*.

4.6 APPLICATIONS

Fourier analysis finds engineering applications in circuit analysis, filtering, amplitude modulation, rectification, harmonic distortion, and signal transmission through a linear system. We will only consider three simple applications: circuit analysis, spectrum analyzers, and filters.

4.6.1 Circuit Analysis

Fourier analysis enables us to find the steady response of a circuit to a periodic excitation.

The first step involves finding the Fourier series expansion of the excitation. This Fourier series representation may be regarded as a series-connected sinusoidal sources with each source having its own amplitude and frequency. The second step is finding the response to each term in the Fourier series by the phasor techniques. Finally, following the principle of superposition, we add all the individual responses.

Example 4.8

If the signal $g(t)$ in Figure 4.4 (see Example 4.2) is the voltage source $v_s(t)$ in the circuit of Figure 4.16, find the response $v_o(t)$ of the circuit.

SOLUTION

From Example 4.2,

$$v_s(t) = 2.5 + \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi t)$$

where $T = 2$, $\omega_o = 2\pi/T = \pi$, $\omega_n = n\omega_0 = n\pi$ rad/second. Using phasors, we obtain the response $\boldsymbol{V}_o$ in the circuit of Figure 4.16 by voltage division; i.e.,

$$\boldsymbol{V}_o = \frac{1/j\omega_n C}{R + 1/j\omega_n C}\boldsymbol{V}_s = \frac{1}{1 + j\omega_n RC}\boldsymbol{V}_s = \frac{\boldsymbol{V}_s}{1 + j4n\pi} \tag{4.8.1}$$

where $R = 4$, $C = 1$. For the DC component ($\omega_n = 0$ or $n = 0$)

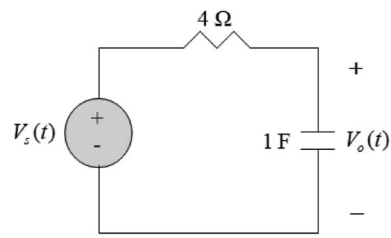


FIGURE 4.16 For Example 4.8.

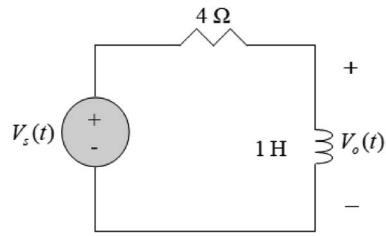


FIGURE 4.17 For Practice Problem 4.8.

$$\boldsymbol{V}_s = 2.5 \Rightarrow \boldsymbol{V}_o = 2.5$$

This is expected since the capacitor is an open circuit to DC. For the *n*th harmonic,

$$\boldsymbol{V}_s = \frac{5}{n\pi}\angle 0° \tag{4.8.2}$$

and the corresponding response is obtained from eq. (4.8.1) as

$$\begin{aligned}\boldsymbol{V}_o &= \frac{1}{\sqrt{1+16n^2\pi^2}\angle \tan^{-1} 4n\pi}\left(\frac{5}{n\pi}\angle 0°\right)\\ &= \frac{5\angle(-\tan^{-1} 4n\pi)}{n\pi\sqrt{1+16n^2\pi^2}}\end{aligned} \tag{4.8.3}$$

In the time domain,

$$v_o(t) = 2.5 + \sum_{n=1}^{\infty} \frac{5}{n\pi\sqrt{1+16n^2\pi^2}} \sin\left(n\pi t - \tan^{-1} 4n\pi\right), \tag{4.8.4}$$

Practice Problem 4.8 Let the signal in Figure 4.3 (see Practice Problem 4.1) be applied to the circuit in Figure 4.17. Find the response $v_o(t)$.

Answer: $v_o(t) = \sum_{n=1}^{\infty} \frac{6}{\sqrt{9+n^2\pi^2}} \cos(n\pi t - \tan^{-1} n\pi/3)$.

4.6.2 Spectrum Analyzers

As we have observed, the Fourier series provides the spectrum of a signal. By providing the spectrum of a signal $x(t)$, the Fourier series helps us identify which frequencies are playing an important role in the shape of the output and which ones are not. For example, audible sounds have significant components in the frequency range of 20 Hz to 15 kHz, while microwaves range from 2.4 GHz to 300 GHz.

We regard a periodic signal as *band-limited* if its amplitude spectrum contains only a finite number of coefficients A_n or c_n. For a band-limited signal, the Fourier series becomes (truncated)

$$x(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = a_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t + \phi_n) \tag{4.54}$$

From this, we see that we need only $2N + 1$ terms (namely, a_0, A_1, A_2,…. A_N, ϕ_1, ϕ_2,…, ϕ_N) to completely specify $x(t)$ if ω_0 is known. This leads to the sampling theorem.

The **sampling theorem** states that a band-limited periodic signal whose Fourier series contains N harmonics is uniquely specified by its values at $2N + 1$ instants in one period.

A spectrum analyzer is an instrument that displays the frequency components (spectra lines) of a signal. Spectrum analyzers are commercially available in various sizes and shapes. Figure 4.18 illustrates a typical one. An oscilloscope shows the signal in the time domain, while the spectrum analyzer shows the signal in the frequency domain. The spectrum analyzer is perhaps the most useful instrument for circuit analysis. It is widely used to measure the frequency response, noise and spurious signal analysis, and more.

4.6.3 Filters

Filters are frequency-selective devices used in electronics and communications systems. Here, we investigate how to design filters to select the fundamental component (or any desired harmonic) of the input signal and reject other harmonics. This

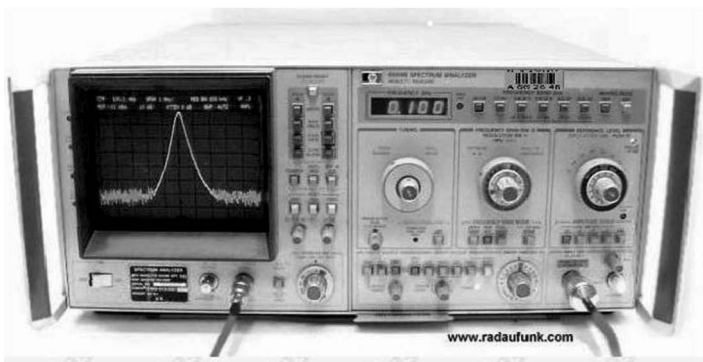


FIGURE 4.18 A spectrum analyzer. (From http://www.radaufunk.com/hp8569b.htm.)

filtering process cannot be accomplished without the Fourier series expansion of the input signal.

A **filter** is a device designed to pass signals with desired frequencies and block or attenuate others.

As shown in Figure 4.19, there are four types of filters:

- A *lowpass filter* (LPF) passes low frequencies and rejects high frequencies.
- A *highpass filter* (HPF) passes high frequencies and rejects low frequencies.
- *A bandpass filter* (BPF) passes frequencies within a frequency band and blocks or attenuates frequencies outside the band.
- *A bandstop filter* (BSF) passes frequencies outside the frequency band and stops or attenuates frequencies within the band.

For the purpose of illustration, we will consider two cases: a lowpass filter and bandpass filter.

A typical lowpass filter is shown in Figure 4.20. The output of a lowpass filter depends on the input signal $v_i(t)$ (with fundamental frequency ω_0), the transfer function $H(\omega)$ of the filter, and the corner or half-power frequency $\omega_c = 1/RC$. The lowpass filter characteristically passes the DC and low-frequency components, while blocking or attenuating the high-frequency components. If we make ω_c sufficiently

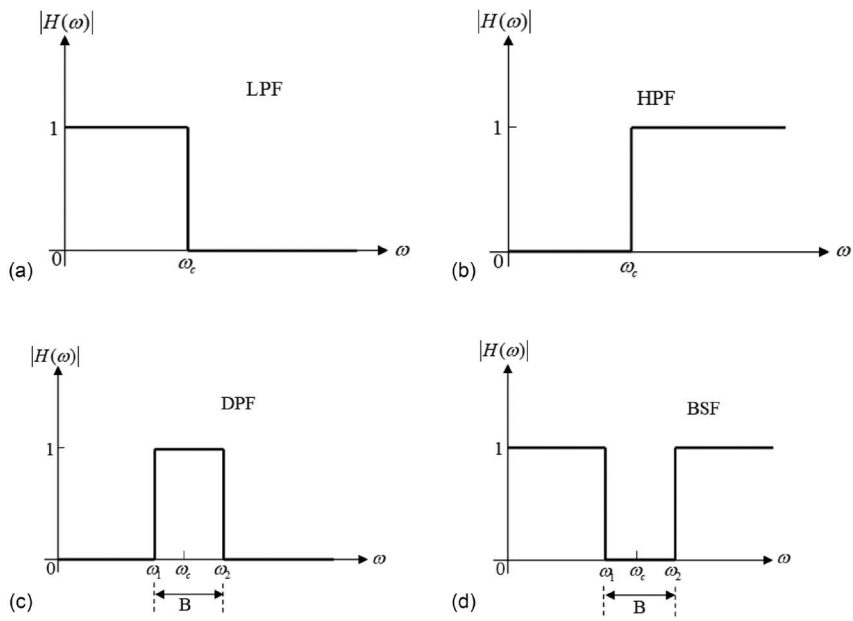


FIGURE 4.19 Ideal frequency response of four types of filters.

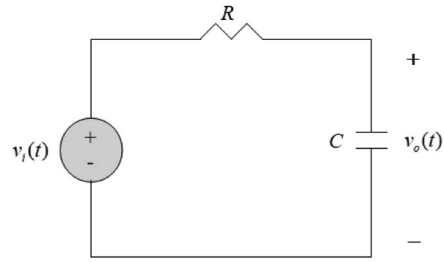


FIGURE 4.20 A typical lowpass filter.

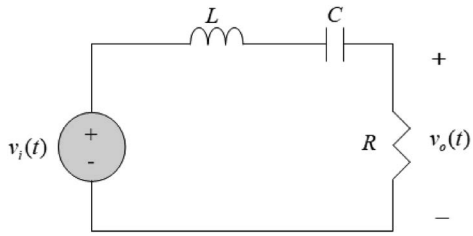


FIGURE 4.21 A typical bandpass filter.

large ($\omega_c >> \omega_0$, e.g., making C small), a large number of the harmonics can be passed. On the other hand, if we make ω sufficiently small ($\omega_c << \omega_0$), we can block out all the AC components and pass only DC.

A typical bandpass filter is shown in Figure 4.21. The output of a bandpass filter depends on the input signal $v_i(t)$ (with fundamental frequency ω_0), the transfer function $H(\omega)$ of the filter, its bandwidth B, and its center frequency ω_c. The filter passes all the harmonics of the input signal within a band of frequencies ($\omega_1 < \omega < \omega_2$) centered around ω_c and rejects other harmonics.

Example 4.9

If the signal in Figure 4.22(a) is applied to an ideal lowpass filter with the transfer function shown in Figure 4.22(b), find the output.

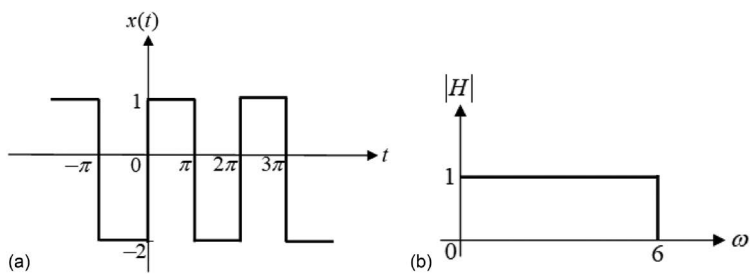


FIGURE 4.22 For Example 4.9.

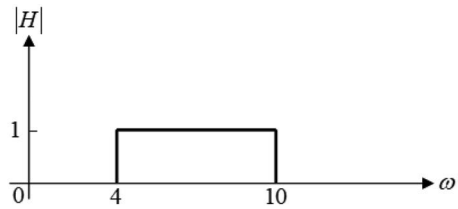


FIGURE 4.23 For Practice Problem 4.9.

SOLUTION

The input signal in Figure 4.22(a) is the same as the signal in Figure 4.3 except the period is 2π instead of 2. Hence $\omega_0 = 2\pi/T = 1$. From Practice Problem 4.1, we know that the Fourier series expansion is (obtained by time scaling)

$$x(t) = -0.5 + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{6}{n\pi} \sin nt$$

or

$$x(t) = -\frac{1}{2} + \frac{6}{\pi}\sin t + \frac{6}{3\pi}\sin 3t + \frac{6}{5\pi}\sin 5t + \frac{6}{7\pi}\sin 7t + \cdots$$

Since the corner frequency of the filter is $\omega_c = 6$ rad/second, only the DC component and harmonics with $n\omega_0 < 6$ will be passed. Hence, the output of the filter is

$$y(t) = -\frac{1}{2} + \frac{6}{\pi}\sin t + \frac{6}{3\pi}\sin 3t + \frac{6}{5\pi}\sin 5t$$

Practice Problem 4.9 Rework Example 4.9 if the ideal lowpass filter is replaced by the ideal bandpass filter whose transfer function is shown in Figure 4.23.

Answer: $y(t) = \frac{6}{5\pi}\sin 5t + \frac{6}{7\pi}\sin 7t + \frac{6}{9\pi}\sin 9t$.

4.7 COMPUTING WITH MATLAB®

Although MATLAB does not help in finding the Fourier series of a signal, it can help in plotting line spectra and examining the convergence of a Fourier series. The following examples show us how.

Example 4.10

Refer to the triangular wave in Figure 4.24. Take $A = 10$ and $T = 2$. Truncate the series at $n = N$. Plot $x(t)$ for $N = 3$, 5, and 10.

Our goal is to examine the convergence of this Fourier series using MATLAB.

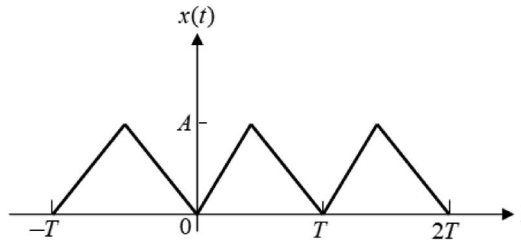


FIGURE 4.24 For Example 4.10.

SOLUTION

We can show that the Fourier series expansion of $x(t)$ is

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\omega_0 t$$

where $A = 10$ and $\omega = 2\pi/T = \pi$. If we truncate the series at $n = N$, then the finite sum becomes

$$x_N(t) = 5 - \frac{40}{\pi^2}\sum_{n=1}^{N}\frac{1}{(2n-1)^2}\cos(2n-1)\pi t$$

The MATLAB script to compute and plot $x_N(t)$ is shown next. Since the period is $T = 2$, we show $x_N(t)$ for only $0 < t < 2$.

```
N = input('Enter the highest harmonic desired'); %3, 5,10
t=0:0.01:2;
xN = 5*ones(1,length(t));  % dc component
fac = -40/(pi*pi);
for n=1:N
xN = xN + fac*cos( (2*n -1)*pi*t )/( (2*n-1)^2);
end
plot(t,xN)
xlabel('t')
ylabel('partial sum')
```

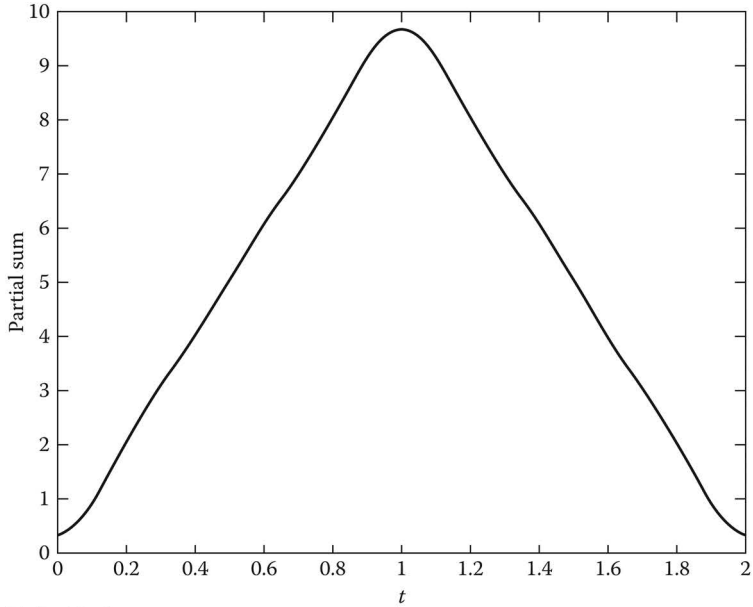
The script was run for $N = 3$ and $N = 5$. The results for these two cases are shown in Figure 4.25. There is a noticeable difference for the two cases. However, for $N = 5$ or greater, the result remains the same. We notice that the Fourier series converges exactly to the signal. This confirms Fourier's theorem, which requires that $x_N(t)$ should converge to $x(t)$ as $N \to \infty$.

Practice Problem 4.10 Repeat Example 4.10 for the periodic square wave shown in Figure 4.26. The Fourier series expansion is

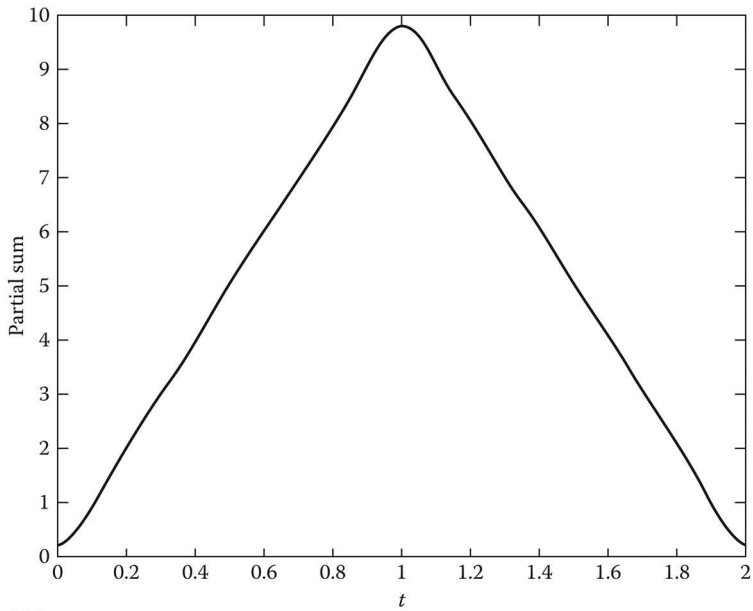
$$x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_0 t$$

Take $A = 10$ and $T = 2$. Try $N = 5$ and $N = 10$.

Answer: See Figure 4.27.



(a) for $N = 3$



(b) for $N = 5$

FIGURE 4.25 For Example 4.12: (a) $x_N(t)$ for $N = 3$; (b) $x_N(t)$ for $N = 5$.

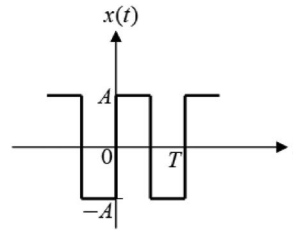
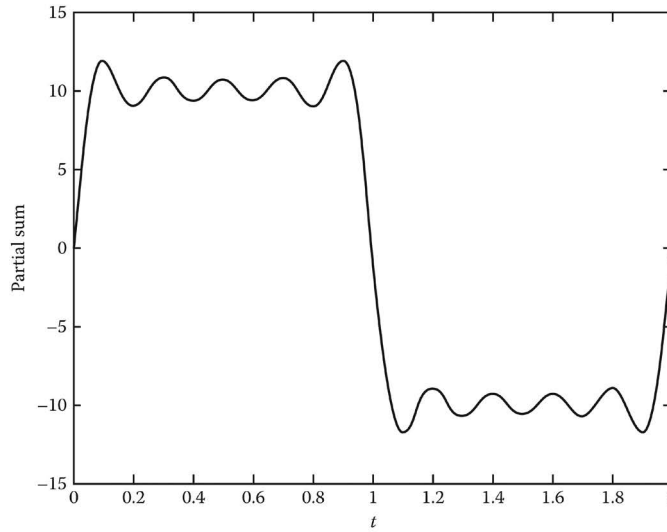


FIGURE 4.26 For Practice Problem 4.10.



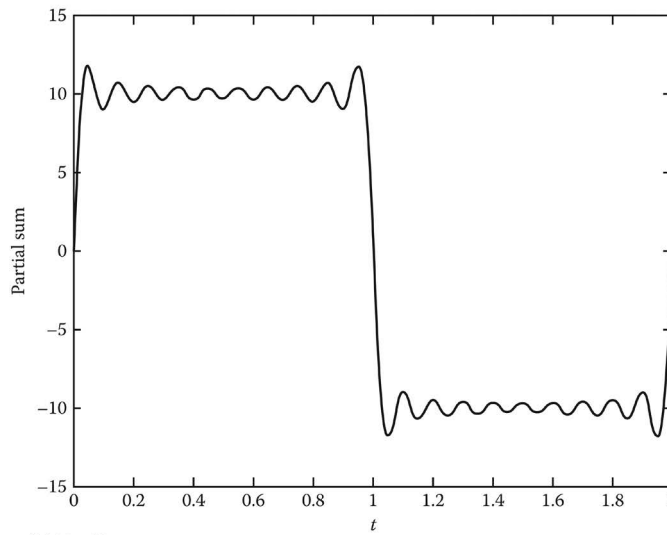


FIGURE 4.27 For Practice Problem 4.12: (a) $x_N(t)$ for $N = 5$, (b) $x_N(t)$ for $N = 10$.

Example 4.11

In Example 4.3, we obtained the exponential Fourier series for $f(t) = t$, $-1 < t < 1$, with $f(t + 2n) = f(t)$ and n is an integer.

$$f(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{j}{n\pi} e^{jn\pi t}, \quad c_0 = 0. \tag{4.11.1}$$

Use MATLAB to plot the line spectrum for the magnitude and the phase of $x(t)$.

SOLUTION

From Eq. (4.11.1), we obtain

$$c_n = (-1)^n \frac{j}{n\pi}, \quad n \neq 0, \quad c_0 = 0. \tag{4.11.2}$$

The MATLAB script shown next uses eq. (4.11.2). The stem plot is shown in Figure 4.28.

```
clear
clc
n=-5:1:5;
c1 = j*(-1).^n;
c2 = n*pi;
if n==0
   cn = 0;
else
cn=c1./c2;
end
mag = abs(cn);
phase = angle(cn)*180/pi;   % converts angle in radians to
degrees
subplot(2,1,1); stem(n,mag)
xlabel('n');  ylabel('|cn|')
subplot(2,1,2); stem(n,phase)
xlabel('n');  ylabel('\theta_n')
```

Practice Problem 4.11 From Practice Problem 4.3, the periodic function $y(t) = t^2$, $0 < t < T$ has the complex Fourier series

$$y(t) = 1.333 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n^2\pi^2}(1 + jn\pi)e^{jn\pi t}$$

Use MATLAB to obtain the magnitude and phase line spectra for the signal $f(t)$.

Answer: See Figure 4.29.

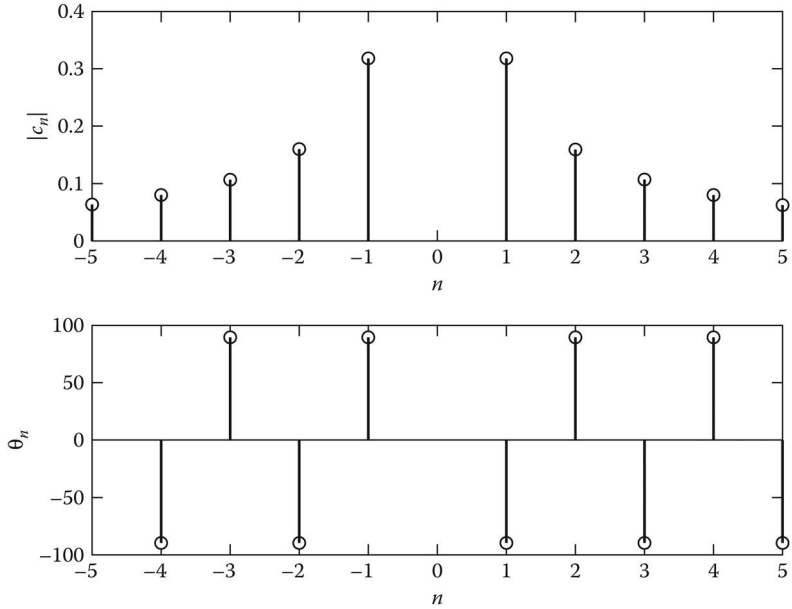


FIGURE 4.28 For Example 4.11.

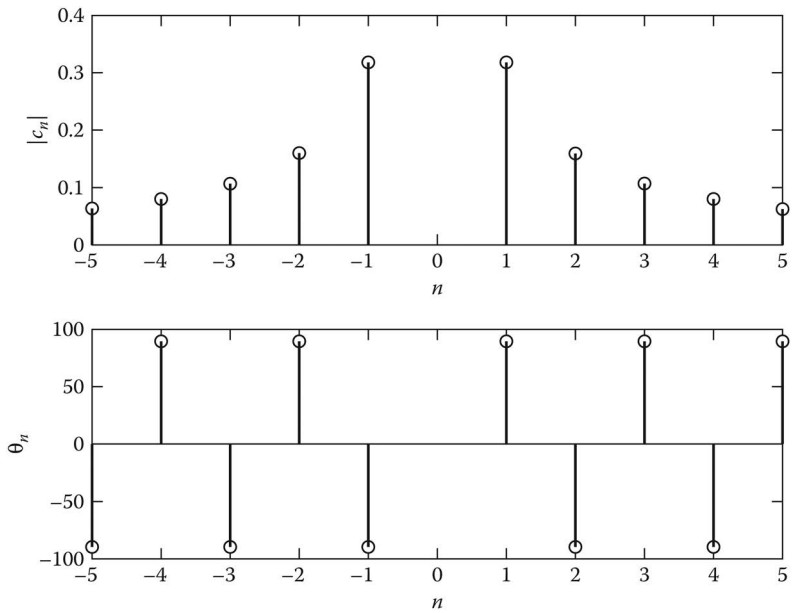


FIGURE 4.29 For Practice Problem 4.11.

SUMMARY

1. A periodic function $x(t)$ repeats itself every period T, i.e., $x(t + T) = x(t)$.
2. A periodic function $x(t)$ can expressed in terms of sinusoids using Fourier series:

$$x(t) = \underbrace{a_0}_{dc} + \underbrace{\sum_{n=1}^{\infty} \left(a_n \cos n\omega_o t + b_n \sin n\omega_o t\right)}_{ac}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency. The Fourier series decomposes the signal into the DC component a_0 and AC components.

There are three basic forms of Fourier series representation: the sine-cosine form, the amplitude-phase form, and the exponential form.

3. The Fourier coefficients of sine-cosine form are determined as:

$$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_o t\, dt$$

$$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_o t\, dt$$

4. The amplitude-phase form of Fourier series is

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t + \phi_n)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

5. The exponential (or complex) form of Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

and $\omega_0 = 2\pi/T$.

6. For a Fourier series to converge, it must meet Dirichlet conditions, which require that the signal $x(t)$ must be integrable, have only a finite number of minima and maxima, and have a finite number of discontinuities over a period.
7. The line spectra (amplitude and phase) of $x(t)$ is the plot of A_n and ϕ_n or $|c_n|$ and θ_n versus frequency.
8. Fourier series representation of a periodic function results in an overshoot behavior at points of discontinuity due to partial summation. The overshoot is about 9% of the peak-to-peak value. This is known as Gibbs phenomenon.
9. The average power of periodic voltage and current is

$$P = \frac{1}{T}\int_0^T |x(t)|^2\,dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2 = \sum_{n=-\infty}^{\infty}|c_n|^2$$

This result is known as *Parseval's theorem*. It states that the average power in a periodic signal is the sum of the average power in its DC component and the average of the powers in its harmonics.

10. In this chapter, Fourier series is applied to circuit analysis, spectrum analyzers, and filters.
11. Fourier series expansion allows us to use the phasor method in analyzing circuits with nonsinusoidal periodic source signal. We apply the phasor technique to determine the response due to each harmonic in the series, transform the responses to the time domain, and add them together.
12. Using MATLAB, we can plot the Fourier series and the line spectra for both magnitude and phase.

REVIEW QUESTIONS

4.1 If $x(t) = 2t$, $0 < t < \pi$, $x(t + T) = x(t + \pi) = x(t)$, the value of ω_0 is
(a) 1, (b) 2, (c) π, (d) 2π

4.2 Which of the following signals are even?
(a) $t^2 \cos t$, (b) $t^2 ln(t)$, (c) $\exp(t^2)$, (d) $t^2 + t^4$, (e) $\sinh t$

4.3 Which of the following signals are odd?
(a) $\sin t$, (b) $t \sin t$, (c) $\exp(t)$, (d) $t^3 \cos t$, (e) $\sinh t$

4.4 If $x(t) = 8\cos 100t + 4\cos 200t + 2\cos 300t + \cos 400t + \cdots$, the signal is:
(a) odd, (b) even, (c) half-wave symmetric, (d) quarter-wave symmetric

4.5 Since $\cos t$ is an even function, $\cos t + \cos(-t)$ is
(a) $2\cos t$, (b) $0.5\cos t$, (c) $2\sin t$, (d) $0.5\sin 2t$

4.6 If $x(t) = 12 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the magnitude of the DC component is
(a) 12, (b) 8, (c) 4, (d) 2, (e) 0

4.7 If $x(t) = 12 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the frequency of the 6th harmonic is
(a) 12, (b) 11, (c) 9, (d) 6

4.8 If $x(t) = 10 + 4\cos t + 2\cos 3t$, the power in the signal is:
(a) 164, (b) 120, (c) 110, (d) 16

4.9 A Fourier cosine series is one that involves only cosine terms.
(a) True, (b) False

4.10 The plot of θ_n versus $n\omega_0$ is called:
(a) complex frequency spectrum, (b) complex phase spectrum, (c) complex amplitude spectrum.

Answer: 4.1 b, 4.2 a, d, 4.3 a, d, e, 4.4 b, 4.5 a, 4.6 a, 4.7 b, 4.8 c, 4.9 a, 4.10 b.

PROBLEMS

Section 4.2: Trigonometric Fourier Series

4.1 Find the trigonometric Fourier series for

$$x(t) = \begin{cases} 5, & 0 < t < \pi \\ 10, & \pi < t < 2\pi \end{cases}$$

$x(t + 2\pi) = x(t)$.

4.2 Obtain the Fourier series of the periodic signal

$$x(t) = 2|t|, \quad -\pi < t < \pi$$

4.3 Determine the sine-cosine Fourier series for the periodic signal in Figure 4.30.

4.4 A periodic signal is given by

$$x(t) = \sum_{n=1}^{\infty} \frac{4}{n} \sin^2\left(\frac{n\pi}{2}\right) \cos(400n\pi t)$$

(a) Compute the period T and the fundamental frequency ω_o.
(b) Identify the Fourier series coefficients a_n and b_n.

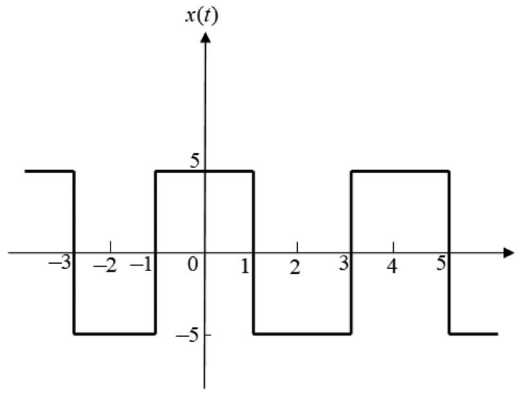


FIGURE 4.30 For Problem 4.3.

4.5 Determine the trigonometric Fourier series of $x(t)$ in Figure 4.31.
4.6 Express the signal in Figure 4.32 in trigonometric Fourier.
4.7 Calculate the trigonometric Fourier series expansion of the signal $y(t)$ in Figure 4.33.
4.8 Obtain the trigonometric Fourier series representation of the signal shown in Figure 4.34.
4.9 Find the trigonometric Fourier series expansion of the signal $y(t)$ in Figure 4.35.
4.10 Determine the trigonometric Fourier series expansion of the signal $z(t)$ in Figure 4.36.
4.11 Obtain the Fourier series expansion for the signal shown in Figure 4.37.
4.12 Calculate the trigonometric Fourier series expansion of the signal $z(t)$ in Figure 4.38.

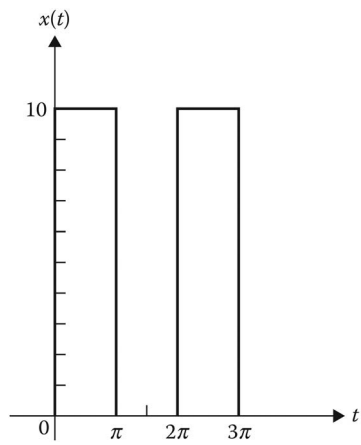


FIGURE 4.31 For Problem 4.5.

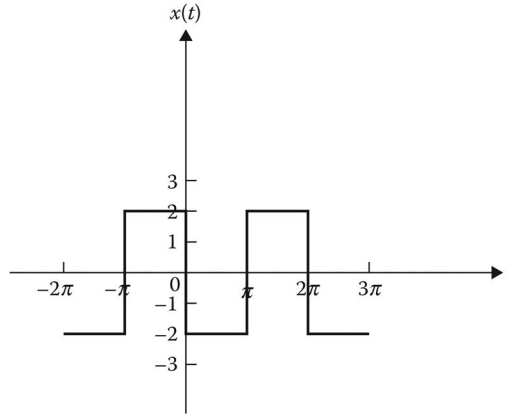


FIGURE 4.32 For Problems 4.6 and 4.17.

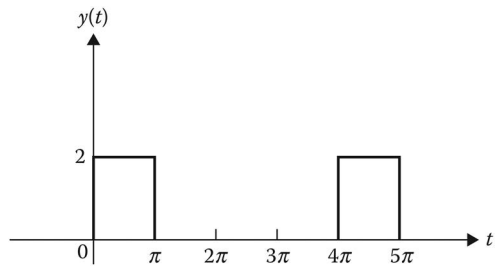


FIGURE 4.33 For Problems 4.7 and 4.18.

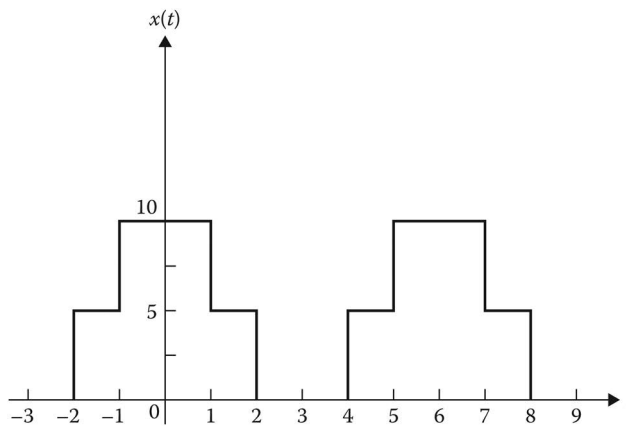


FIGURE 4.34 For Problem 4.8.

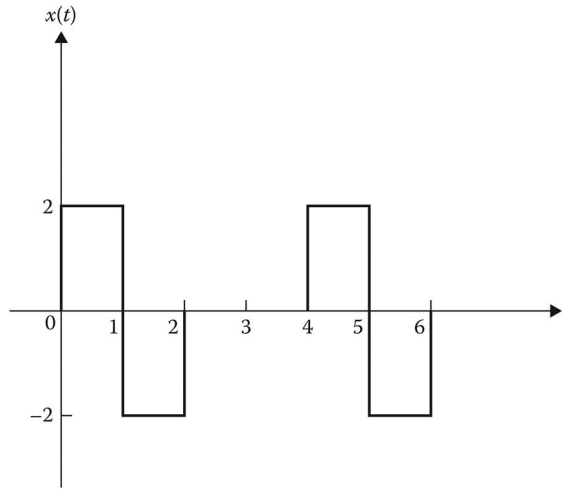


FIGURE 4.35 For Problems 4.9 and 4.20.

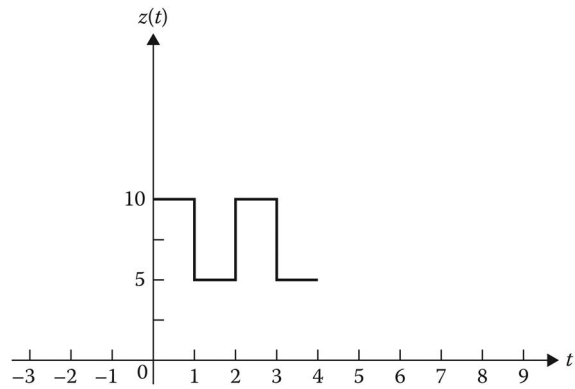


FIGURE 4.36 For Problem 4.10 and 4.21.

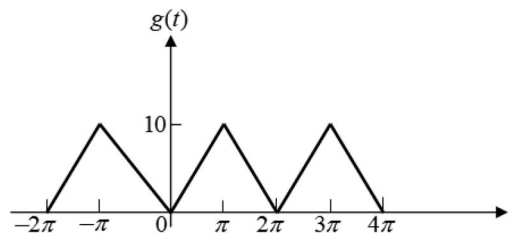


FIGURE 4.37 For Problem 4.11.

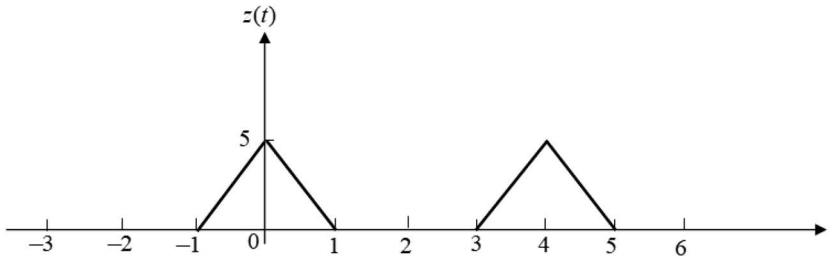


FIGURE 4.38 For Problem 4.12.

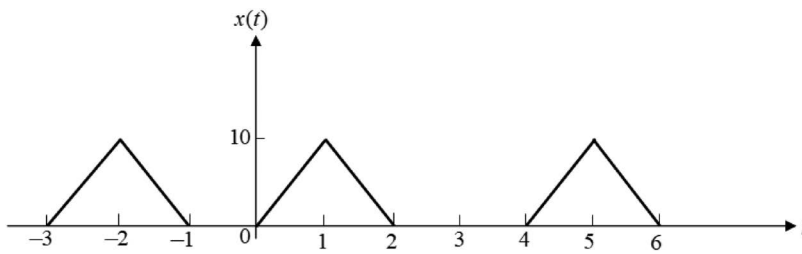


FIGURE 4.39 For Problem 4.13.

4.13 Expand $x(t)$ in Figure 4.39 in Fourier series.

4.14 Plot the amplitude and phase spectra for the signal

$$x(t) = 8\cos(100\pi t - 0.2) + 6\cos(200\pi t - 1.2) - \cos(300\pi t)$$

4.15 Show that

$$A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos\left(\omega t - \tan^{-1}\frac{B}{A}\right)$$

Section 4.3: Exponential Fourier Series

4.16 Obtain the exponential Fourier series coefficients for

$$x(t) = \sin^3(2\pi t)$$

4.17 Express $x(t)$ in Figure 4.32 in exponential Fourier series.

4.18 Determine the exponential Fourier series expansion of $y(t)$ in Figure 4.33.

4.19 Find the exponential Fourier series coefficients of the signal $x(t)$ in Figure 4.40.

4.20 Calculate the exponential Fourier representation of the signal $y(t)$ in Figure 4.35.

4.21 Obtain the exponential Fourier series coefficients of the signal $z(t)$ in Figure 4.36

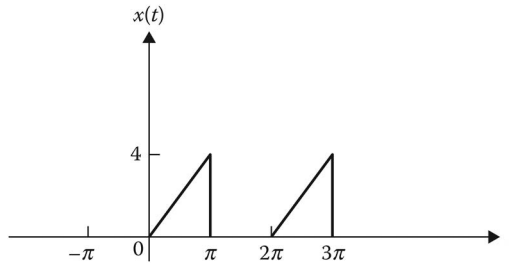


FIGURE 4.40 For Problem 4.19.

4.22 Express the sawtooth waveform $x(t)$ in Figure 4.41 as a complex Fourier series.

4.23 Determine the exponential Fourier series coefficients of the signal in Figure 4.42.

4.24 Determine the exponential Fourier series for the impulse train in Figure 4.43.

4.25 The complex exponential Fourier series of a signal $x(t)$ over $0 < t < T$ is

$$x(t) = \sum_{n=-\infty}^{\infty} j \sin\left(\frac{n\pi}{2}\right) e^{in\pi t/50}$$

(a) Calculate T.
(b) Determine the average value of $x(t)$.
(c) Find the amplitude of the third harmonic.

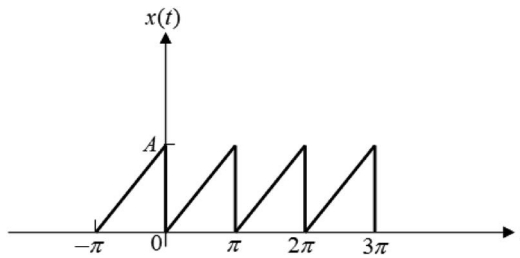


FIGURE 4.41 For Problem 4.22.

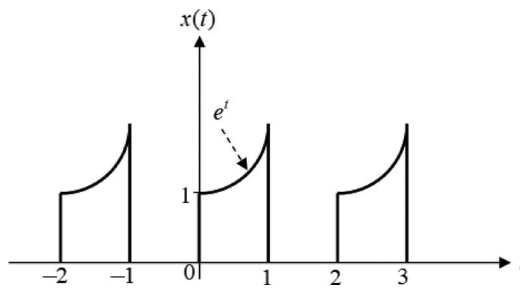


FIGURE 4.42 For Problem 4.23.

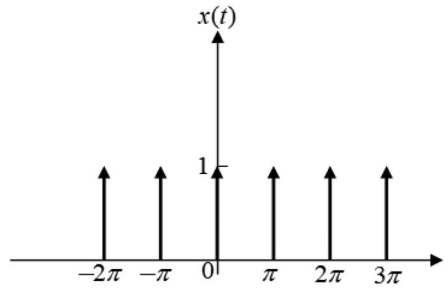


FIGURE 4.43 For Problem 4.24.

4.26 The complex exponential Fourier series of a signal is given as

$$x(t) = (2-j)e^{-j3t} + j4e^{-jt} + 5 - j4e^{jt} + (2+j)e^{j3t}$$

Sketch the amplitude and phase spectra.

4.27 A periodic signal is expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{-jnt}$$

where $c_0 = 0.4$, $c_1 = c_{-1} = 0.2$, $c_2 = c_{-2} = 0.12$, and $c_3 = c_{-3} = 0.08$.

(a) Determine the period T and the fundamental frequency ω_o.

(b) Write out $x(t)$ in exponential form and use Euler relation to express $x(t)$ in cosine form.

4.28 Suppose $x(t) = 1 + \cos 2t + 0.5 \cos 4t$. Find the Fourier series coefficients of the exponential form of $x(t)$.

4.29 Determine the complex Fourier series if $x(t) = 10\cos 3t$.

Section 4.4: Properties of Fourier Series

4.30 A periodic signal is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi/T$ and T is the period.

Determine the Fourier coefficients in terms of c_n for $y(t)$, where:

(a) $y(t) = x(-t)$

(b) $y(t) = x(2t)$

(c) $y(t) = x(t-3)$

(d) $y(t) = \dfrac{dx(t)}{dt}$

4.31 If c_n is the exponential Fourier series with coefficients of a periodic signal $x(t)$, determine the coefficients of the following:
(a) $y(t) = x(3t)$
(b) $z(t) = x(-3t)$
(c) $h(t) = 1 + x(2t)$

4.32 Let $w(t) = x(t)y(t)$. Show that if $x(t)$ is even and $y(t)$ is odd, $w(t)$ is odd.

4.33 Let $x(t)$ and $y(t)$ be periodic signals with the same period T such that

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}, \quad y(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega_o t},$$

where $\omega_o = 2\pi/T$. Show that the signal $z(t) = x(t)y(t)$ is periodic with the same period T and can be expressed as

$$z(t) = \sum_{n=-\infty}^{\infty} e_n e^{jn\omega_o t}$$

where $e_n = \sum_{m=-\infty}^{\infty} c_m d_{n-m}$

4.34 Consider the signals shown in Figure 4.44. Which of them have (a) even symmetry? (b) odd symmetry? (c) half-wave odd symmetry?

4.35 Without evaluating any Fourier series coefficients, state which of the signals in Figure 4.44 contain (a) a DC component, (b) only odd harmonics, (c) both sine and cosine terms.

4.36 Given the exponential Fourier series of a periodic signal as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

Show that

$$P = \frac{1}{T}\int_0^T |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

4.37 The first half cycle of a periodic signal is shown in Figure 4.45. Sketch the signal over $-T/2 < t < T/2$ for each of these cases: (a) $y(t)$ is even, (b) $y(t)$ is odd.

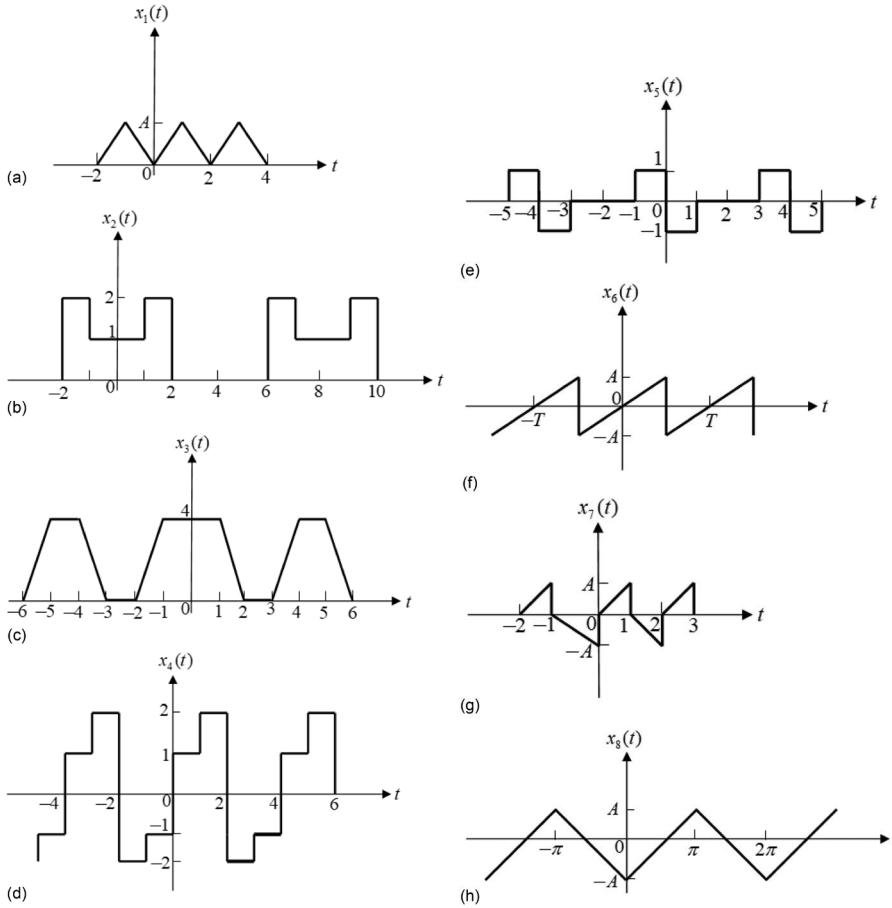


FIGURE 4.44 For Problems 4.34 and 4.35.

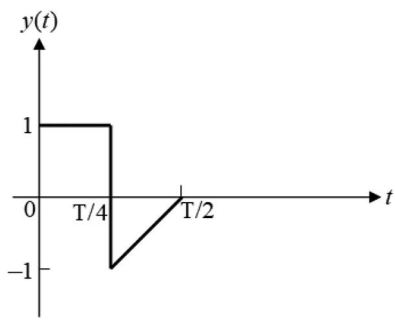


FIGURE 4.45 For Problem 4.37.

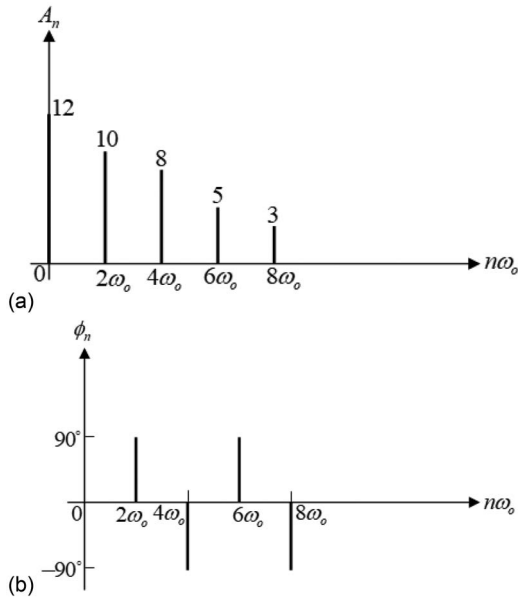


FIGURE 4.46 For Problem 4.40.

4.38 Calculate the average power dissipated in a 4-Ω resistor when applied to the current

$$i(t) = 8 + 6\cos \omega t - 4\cos 2\omega t + 2\cos 3\omega t \text{ A}$$

4.39 Determine the average power dissipated in a 3-Ω resistor when applied to the current

$$i(t) = 8\cos(2\pi t) + 6\sin(2\pi t) - 5\cos(4\pi t) - 3\sin(4\pi t) + \cos(6\pi t) \text{ A}$$

4.40 The amplitude and phase spectra of a truncated Fourier series are shown in Figure 4.46.
(a) Find an expression for the periodic voltage using amplitude-phase form.
(b) Is the voltage an odd or even function of t?

4.41 The exponential Fourier coefficients of the signal in Figure 4.47(a) is:

$$c_n = \begin{cases} \dfrac{A\sin^2(n\pi/2)}{(n\pi/2)^2}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

Find the coefficients for the signal in Figure 4.47(b).

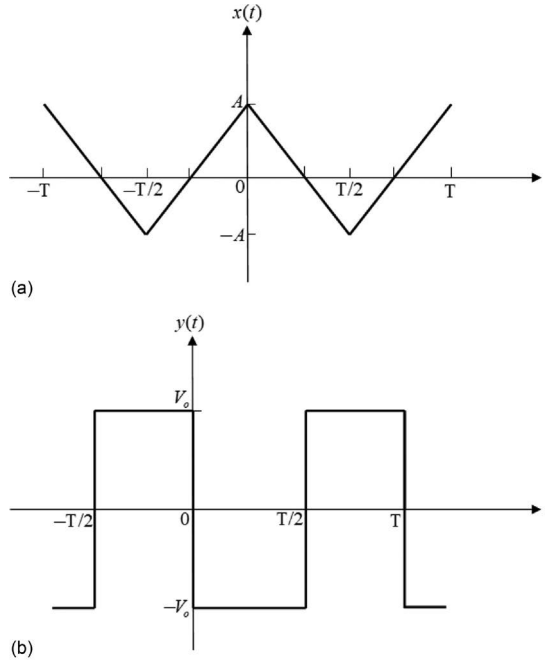


FIGURE 4.47 For Problem 4.41.

Section 4.5: Applications

4.42 Let the input voltage $x(t)$ to the circuit in Figure 4.48 be

$$x(t) = 1 + 4(\cos t + \cos 2t + \cos 3t + \cos 4t).$$

Determine the output voltage $y(t)$.

4.43 A band-limited signal has the following Fourier series representation:

$$i_s(t) = 8 + 6\cos(2\pi t + 30^o) + 4\cos(4\pi t - 150^o)\ \text{mA}$$

If the signal is applied to the circuit in Figure 4.49, find $v(t)$.

4.44 An RLC circuit is shown in Figure 4.50. Find $v_o(t)$ if

$$v_s(t) = 2 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(2n\pi t)$$

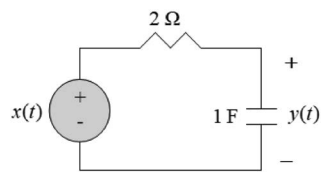


FIGURE 4.48 For Problem 4.42.

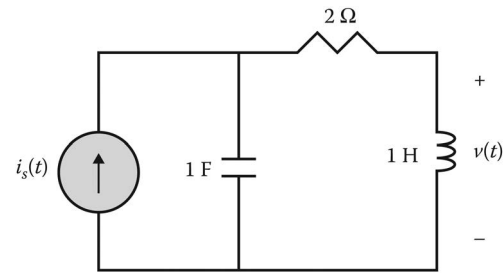


FIGURE 4.49 For Problem 4.43.

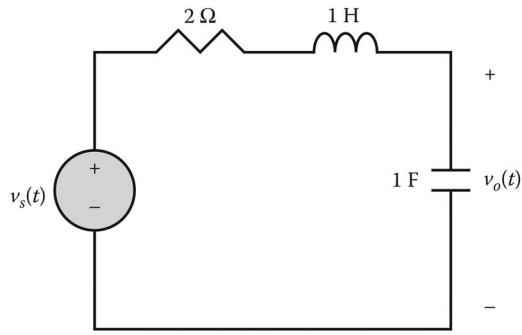


FIGURE 4.50 For Problem 4.44.

4.45 Determine $v_o(t)$ in Figure 4.51. Assume that

$$v_s(t) = 4 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin(2n\pi t)$$

4.46 The periodic current waveform in Figure 4.52 is applied across a 2-kΩ resistor. Find the percentage of the total average power dissipation caused by the DC component.

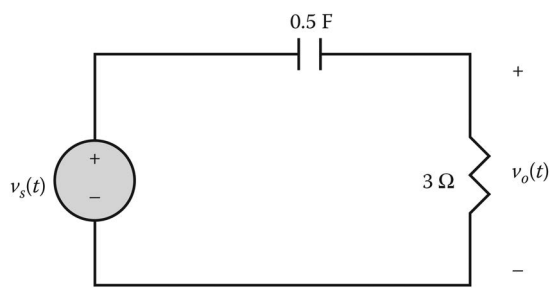


FIGURE 4.51 For Problem 4.45.

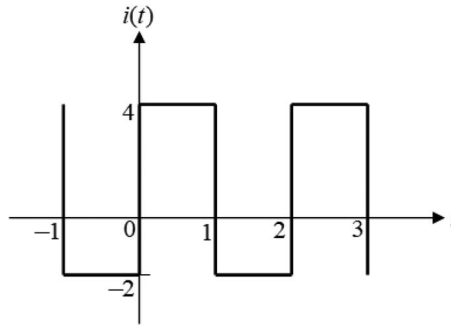


FIGURE 4.52 For Problem 4.46.

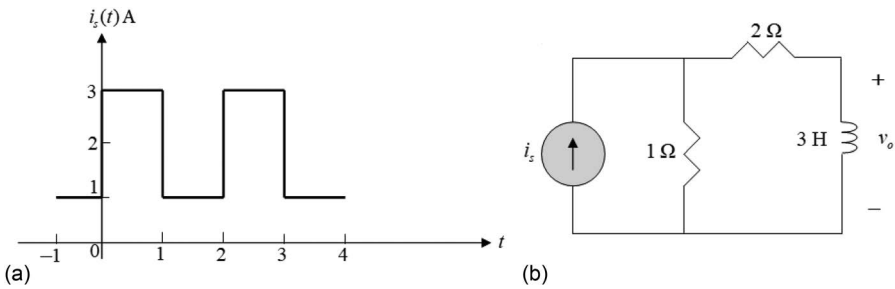


FIGURE 4.53 For Problem 4.47.

4.47 If the periodic current waveform in Figure 4.53 (a) is applied to the circuit in Figure 4.53(b), find v_o.

4.48 Find the response $i_o(t)$ for the circuit in Figure 4.54(a), where $v(t)$ is shown in Figure 4.54(b).

4.49 If the periodic current signal in Figure 4.55 (a) is applied to the circuit in Figure 4.55(b), determine $i_o(t)$.

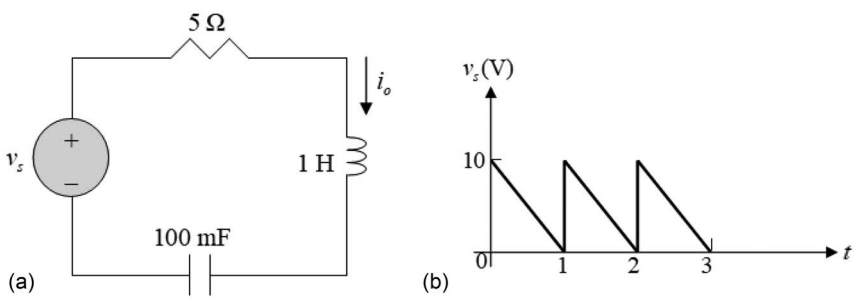


FIGURE 4.54 For Problem 4.48.

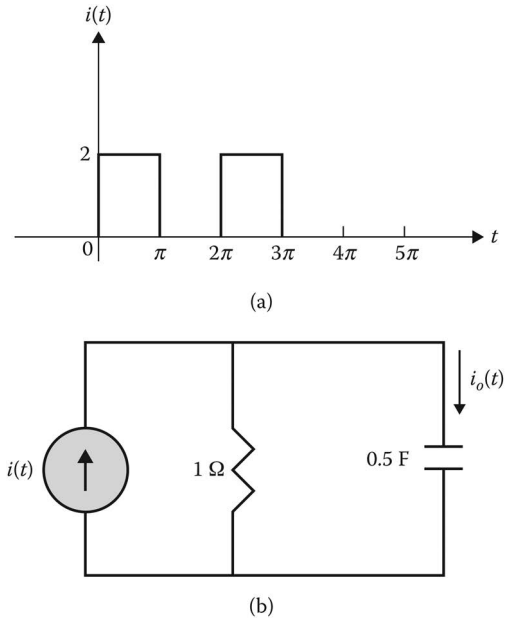


FIGURE 4.55 For Problem 4.49.

Section 4.6: Computing with MATLAB

4.50 Using MATLAB, write a program to compute and plot the partial sum of the sawtooth waveform in Figure 4.4.

4.51 Use MATLAB to plot:

(a) $x(t) = 10 + 6\cos 2\pi t + 4\cos 4\pi t,\ 0 < t < 2$

(b) $y(t) = \sum_{n=-5}^{5} \frac{(-1)^n}{10} \sin\left(\frac{n\pi t}{2}\right), -1/2 < t < 1/2$

Use MATLAB to plot the partial sum $x_N(t)$ for $N = 5$ and $N = 9$.

4.52 Use MATLAB to plot the Fourier series expansion

$$x(t) = \frac{10}{\pi^2}\left[\sin t - \frac{1}{9}\sin 3t + \frac{1}{25}\sin 5t - \frac{1}{49}\sin 7t\right]$$

4.53 The exponential Fourier series coefficient of a signal is

$$c_n = -\frac{1}{j2n\pi}(\cos n\pi - 1)$$

Using MATLAB, plot the magnitude and phase spectra. Take $-5 < n < 5$.

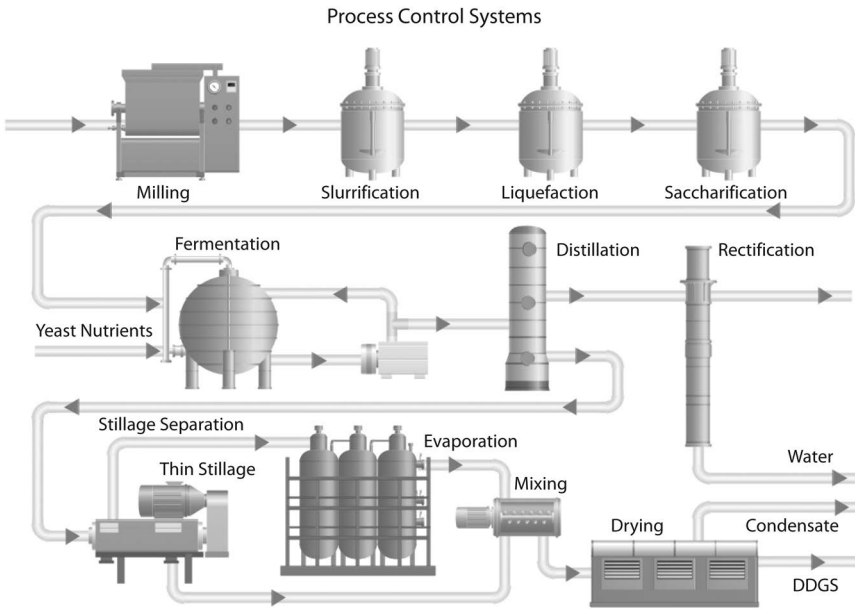
4.54 Repeat the previous problem for

$$c_n = \frac{j10}{n\pi}\left((-1)^n - 1\right)$$

5 The Fourier Transform

We can't solve problems by using the same kind of thinking we used when we created them.

Albert Einstein



Control systems are an area of electrical engineering where signals and systems are used. A control system is designed to regulate the behavior of one or more variables in some desired manner. Control systems play major roles in our everyday life. Household appliances such as heating and air-conditioning systems, switch-controlled thermostats, washers and dryers, cruise controllers in automobiles, elevators, traffic lights, manufacturing plants, navigation systems—all utilize control systems. In the aerospace field, precision guidance of space probes, the wide range of operational modes of the space shuttle, and the ability to maneuver space vehicles remotely from earth all require knowledge of control systems. In the manufacturing sector, repetitive production line operations are increasingly performed by robots, which are programmable control systems designed to operate for many hours without fatigue.

Control engineering applies signals and systems theory. It is not limited to any specific engineering discipline but may involve environmental, chemical, aeronautical, mechanical, civil, and electrical engineering. For example, a typical task

DOI: 10.1201/9781003490869-5

for a control engineer might be to design a speed regulator for a disk drive head. A thorough understanding of control systems techniques is essential to the electrical engineer and is of great value for designing control systems to perform the desired task.

5.1 INTRODUCTION

In Chapter 4, we saw that Fourier series enabled us to represent a periodic signal as a linear combination of infinite sinusoids. The Fourier series expansion reveals the frequency content of the periodic signal. It is also possible to analyze the frequency content of nonperiodic signals. The tool that enables us to do this is the Fourier transform (also known as the Fourier integral). The transform assumes that a nonperiodic (or aperiodic) signal is a periodic signal with an infinite period.

The **Fourier transform** is a mathematical procedure that converts a time-domain signal into the same signal expressed in the frequency domain.

It provides another way to represent a signal. The Fourier transform is important in analyzing and processing signals in science and engineering, especially in image processing, medical images, computerized axis tomography (CAT), and magnetic resonance imaging (MRI).

The Fourier transform is similar to the Laplace transform. Both are integral transforms. The Fourier transform may be regarded as a special case of the Laplace transform with $s = j\omega$, when the Laplace transform exists. Just like the bilateral Laplace transform, the Fourier transform can deal with systems having inputs for $t < 0$ as well as those for $t > 0$. The Fourier transform is very useful in linear systems, filtering, communication systems, and digital signal processing, in situations where the Laplace transform does not exist.

The chapter begins by defining the Fourier transform from Fourier series. Then we present and prove the properties of the Fourier transform, which are useful in deriving Fourier transform pairs. We discuss Parseval's theorem, compare the Laplace and Fourier transforms, and see how the Fourier transform is applied in circuit analysis, amplitude modulation, and sampling. We finally learn how to use MATLAB® in Fourier analysis.

5.2 DEFINITION OF THE FOURIER TRANSFORM

There is a close connection between the definition of Fourier series and the Fourier transform for functions $x(t)$ that are zero outside of an interval. For such a function, we can calculate its Fourier series on any interval that includes the points where $x(t)$ is not identically zero. The Fourier transform is also defined for such a function. As we increase the length of the interval on which we calculate the Fourier

series, then the Fourier series coefficients begin to look like the Fourier transform, and the sum of the Fourier series of $x(t)$ begins to look like the inverse Fourier transform.

The Fourier transform is essentially a generalization of the complex Fourier series in the limit as $T \to \infty$. Recall from eq. (4.15) that if $x(t)$ is a periodic function with period T, then we can expand it in a complex Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{5.1}$$

where

$$c_n = \frac{1}{T}\int_0^T x(t) e^{-n\omega_0 t}\, dt \tag{5.2}$$

Eq. (5.2) can be written as

$$c_n T = \int_0^T x(t) e^{-n\omega_0 t}\, dt \tag{5.3}$$

If we let $T \to \infty$, the fundamental frequency, $\omega_0 = 2\pi/T$ becomes extremely small and the quantity $n\omega_0$ becomes a continuous quantity that can take on any value. We define a new variable $\omega = n\omega_0$ and let $X(\omega) = Tc_n$. Making these substitutions in eq. (5.2) leads to the Fourier transform:

$$X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{\infty}^{-\infty} x(t)\, e^{-j\omega t}\, dt \tag{5.4}$$

where $\mathcal{F}$ is the Fourier transform operator.

> The **Fourier transform** of a signal $x(t)$ is the integration of the product of $x(t)$ and $e^{-j\omega t}$ over the interval from $-\infty$ to $+\infty$.

$X(\omega)$ is an integral transformation of $x(t)$ from the time domain to the frequency domain and is generally a complex function. $X(\omega)$ is known as the spectrum of $x(t)$. The plot of its magnitude $|X(\omega)|$ versus ω is called the amplitude spectrum, while that of its phase $\angle X(\omega)$versus ω is called the phase spectrum. If a signal is real-valued, its magnitude spectrum is even, i.e., $|X(\omega)| = |X(-\omega)|$ and its phase spectrum is odd, i.e., $\angle X(\omega) = -\angle X(-\omega)$. Both the amplitude and phase spectra provide the physical interpretation of the Fourier transform.

In the same manner, we can derive the inverse Fourier transform by multiplying and dividing eq. (5.1) by T.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} T c_n e^{jn\omega_0 t} \frac{1}{T} \tag{5.5}$$

Again, as $T \to \infty$, the fundamental frequency, $\omega_0 = 2\pi/T$, becomes extremely small. Replacing the sum by an integral yields

$$x(t) = \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega t} \frac{d\omega}{2\pi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{5.6}$$

From this, we obtain the inverse Fourier transform as

$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \tag{5.7}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform operator. We say that the signal $x(t)$ and its transform $X(\omega)$ form a Fourier transform pair and denote their relationship by:

$$x(t) \Leftrightarrow X(\omega) \tag{5.8}$$

We will denote signals by lowercase letters, while their transforms are shown in uppercase letters.

Not all functions have Fourier transforms. The necessary conditions required for a function to be transformable (or that the Fourier transform exists) are called the Dirichlet conditions, similar to the Fourier series Dirichlet conditions. These are listed as:

1. $x(t)$ is bounded.
2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.
4. $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \tag{5.9}$$

Therefore, absolutely integrable signals that are continuous or that have a finite number of discontinuities have Fourier transforms.

Example 5.1

Find the Fourier transform of the following functions: (a) $\delta(t)$, (b) $e^{j\omega_0 t}$, (c) $\sin\omega_0 t$, (d) $e^{-at}u(t)$.

SOLUTION

a. For the impulse function,

$$X(\omega) = \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1 \tag{5.1.1}$$

where the sifting property of the impulse function in eq. (1.21) has been applied. Thus,

$$\mathcal{F}[\delta(t)] = 1 \tag{5.1.2}$$

This shows that the magnitude of the spectrum of the impulse function is constant; that is, all frequencies are equally represented in the impulse function.

b. From eq. (5.1.2)

$$\delta(t) = \mathcal{F}^{-1}[1]$$

Using the inverse Fourier transform formula in eq. (5.7),

$$\delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1 e^{j\omega t}\, d\omega$$

or

$$\int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = 2\pi\delta(t) \tag{5.1.3}$$

Interchanging variables t and ω results in

$$\int_{-\infty}^{\infty} e^{j\omega t}\, dt = 2\pi\delta(\omega) \tag{5.1.4}$$

Using this result, the Fourier transform of the given function is

$$\mathcal{F}[e^{j\omega_o t}] = \int_{-\infty}^{\infty} e^{j\omega_o t}\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} e^{j(\omega_o - \omega)t} dt = 2\pi\delta(\omega_0 - \omega)$$

Since the impulse function is an even function, $\delta(\omega_0 - \omega) = \delta(\omega - \omega_0)$,

$$\mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0) \tag{5.1.5}$$

By simply changing the sign of ω_0, we readily obtain

$$\mathcal{F}[e^{-j\omega_0 t}] = 2\pi\delta(\omega + \omega_0) \tag{5.1.6}$$

Also, by setting $\omega_0 = 0$,

$$\mathcal{F}[1] = 2\pi\delta(\omega) \tag{5.1.7}$$

c. By using the result in eqs. (5.1.5) and (5.1.6), we get

$$\begin{aligned} \mathcal{F}[\sin\omega_0 t] &= \mathcal{F}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2j}\right] \\ &= \frac{1}{2j}\mathcal{F}[e^{j\omega_0 t}] - \frac{1}{2j}\mathcal{F}[e^{-j\omega_0 t}] \\ &= j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \end{aligned} \tag{5.1.8}$$

d. Let $x(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t < 0 \end{cases}$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt \\ \mathcal{F}\left[e^{-at}u(t)\right] = X(\omega) &= \frac{-1}{a + j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a + j\omega} \end{aligned} \tag{5.1.9}$$

Practice Problem 5.1 Determine the Fourier transform of the following functions: (a) the rectangular pulse $\Pi(t/\tau)$, (b) $\delta(t + 3)$, (c) $2\cos\omega_0 t$.

Answer: (a) $\frac{2}{\omega}\sin\frac{\omega\tau}{2} = \tau\,\mathrm{sinc}\frac{\omega\tau}{2}$, (b) $e^{j3\omega}$, (c) $2\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$.

Example 5.2

Obtain the Fourier transform of the signal shown in Figure 5.1.

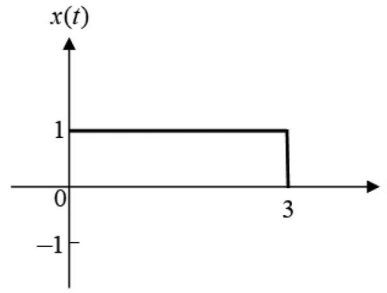


FIGURE 5.1 For Example 5.2.

SOLUTION

This can be done in three ways.

Method 1: Directly using the definition of Fourier transform.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{0}^{3} (1)e^{-j\omega t}\,dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t}\Big|_{0}^{3} = \frac{1}{j\omega}(1 - e^{-j3\omega})$$

Method 2: Expressing $x(t)$ in terms of unit step function.

$$x(t) = u(t) - u(t-3)$$

Taking the Fourier transform of each term gives

$$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega} - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)e^{-j3\omega} = \frac{1}{j\omega}(1 - e^{-j3\omega})$$

Method 3: Using the derivative of $x(t)$.

Take the derivative of $x(t)$ to obtain

$$x'(t) = \delta(t) - \delta(t-3)$$

Take the Fourier transform of each term.

$$j\omega X(\omega) = 1 - (1)e^{-j3\omega}$$

or

$$X(\omega) = \frac{1}{j\omega}(1 - e^{-j3\omega})$$

The Fourier transform property used here is covered later in Section 5.3.3.

Practice Problem 5.2 Derive the Fourier transform of $y(t) = u(t-2) - u(t-5)$.

Answer: $\frac{1}{j\omega}\left(e^{-j2\omega} - e^{-5\omega}\right)$

Example 5.3

Find the Fourier transform of the two-sided exponential pulse shown in Figure 5.2. Sketch the transform.

SOLUTION

Let $x(t) = e^{-a|t|} = \begin{cases} e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}$

The Fourier transform is

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$

$X(\omega)$ is real in this case, and it is sketched in Figure 5.3.

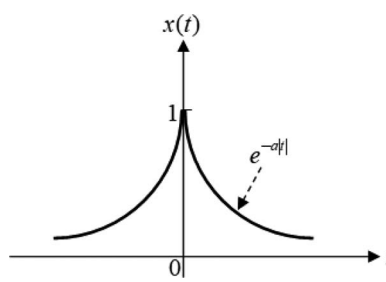


FIGURE 5.2 For Example 5.3.

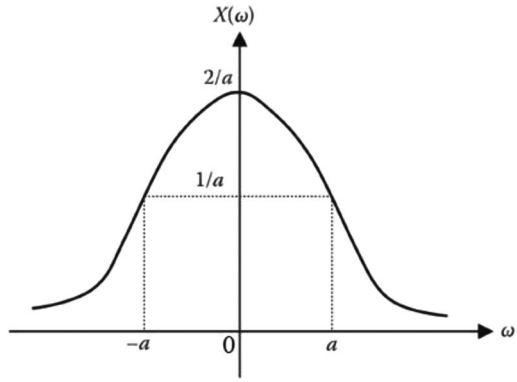


FIGURE 5.3 Fourier transform of $x(t)$ in Figure 5.2.

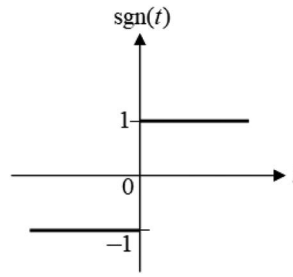


FIGURE 5.4 The signum function of Practice Problem 5.3.

Practice Problem 5.3 Determine the Fourier transform of the signum function in Figure 5.4, i.e., $x(t) = \text{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$. Sketch $|X(\omega)|$.

Answer: $\frac{2}{j\omega}$. See Figure 5.5.

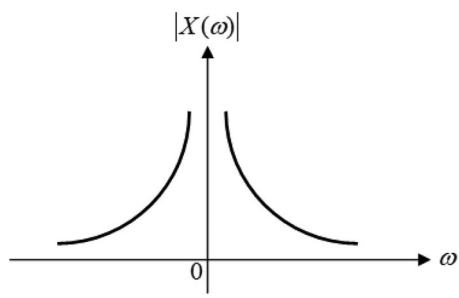


FIGURE 5.5 Fourier transform of the signal in Figure 5.4.

5.3 PROPERTIES OF FOURIER TRANSFORM

The properties of the Fourier transform are summarized next. These properties have important interpretations and meaning in the context of signals and signal processing. The properties of the Fourier transform provide valuable insight into how signal operating in the time domain is described in the frequency domain. Many properties like linearity, time scaling and shifting, frequency shifting, time differentiation and integration, frequency differentiation, duality, convolution, and Parseval's theorem are discussed next.

5.3.1 Linearity

The Fourier transform is a linear transform. That is, suppose we have two functions $x_1(t)$ and $x_2(t)$, with Fourier transform given by $X_1(\omega)$ and $X_2(\omega)$, respectively; then the Fourier transform of $x_1(t)$ and $x_2(t)$ can be easily found as

$$\mathcal{F}\left[a_1x_1(t)+a_2x_2(t)\right]=a_1X_1(\omega)+a_2X_2(\omega) \tag{5.10}$$

where a_1 and a_2 are constants (real or complex numbers). Eq. (5.10) can be easily shown to be true by using the definition of the Fourier transform:

$$\begin{aligned}\mathcal{F}\left[a_1x_1(t)+a_2x_2(t)\right]&=\int_{-\infty}^{\infty}\left[a_1x_1(t)+a_2x_2(t)\right]e^{-j\omega t}\,dt\\&=\int_{-\infty}^{\infty}a_1x_1(t)\,e^{-j\omega t}\,dt+\int_{-\infty}^{\infty}a_2x_2(t)\,e^{-j\omega t}\,dt\\&=a_1X_1(\omega)+a_2X_2(\omega)\end{aligned} \tag{5.11}$$

This can be extended to a linear combination of an arbitrary number of signals.

5.3.2 Time Scaling

The scaling property of the Fourier transformation is as follows. If $X(\omega)=\mathcal{F}\left[x(t)\right]$ and a is a real constant. Then

$$\mathcal{F}\left[x(at)\right]=\frac{1}{|a|}X\left(\frac{\omega}{a}\right) \tag{5.12}$$

resulting in a new frequency ω/*a*. Eq. (5.12) can be found using the definition:

$$\mathcal{F}\left[x(at)\right] = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt$$

Letting $\lambda = at$, $d\lambda = a\,dt$ so that

$$\mathcal{F}\left[x(at)\right] = \int_{-\infty}^{\infty} x(\lambda)e^{-\frac{j\omega\lambda}{a}}\,\frac{d\lambda}{a} = \frac{1}{a}X\left(\frac{\omega}{a}\right) \tag{5.13}$$

Now if a is positive $(a > 0)$,

$$\mathcal{F}\left[x(at)\right] = \int_{-\infty}^{\infty} x(\lambda)e^{-\frac{j\omega\lambda}{a}}\,\frac{d\lambda}{a} = \frac{1}{a}X\left(\frac{\omega}{a}\right) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

If a is negative $(a < 0)$,

$$\mathcal{F}\left[x(at)\right] = \int_{\infty}^{-\infty} x(\lambda)e^{-\frac{j\omega\lambda}{a}}\,\frac{d\lambda}{a} = -\int_{-\infty}^{\infty} x(\lambda)e^{-\frac{j\omega\lambda}{a}}\,\frac{d\lambda}{a} = \frac{1}{-a}X\left(\frac{\omega}{a}\right) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Therefore, we can see that for the general case of scaling with a real number "*a*" we get eq. (5.12). The scaling property is sometimes also called the reciprocal spreading property.

5.3.3 Time Shifting

Another simple property of the Fourier transform is time shifting. If $X(\omega) = \mathcal{F}\left[x(t)\right]$ and t_o is a constant, then

$$\mathcal{F}\left[x(t-t_0)\right] = e^{-j\omega t_0}\,X(\omega) \tag{5.14}$$

If the original function $x(t)$ is shifted in time by a constant amount, it should have the same magnitude of the spectrum, $X(\omega)$. That is, a time delay does not cause the frequency content of $X(\omega)$ to change at all. Since the complex exponential always has a magnitude of 1, we see that the time delay alters the phase of $X(\omega)$ but not its magnitude.

By definition,

$$\mathcal{F}\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} \left[x(t-t_0)\right]e^{-j\omega t}\,dt \tag{5.15}$$

Letting $\lambda = t - t_o$, $d\lambda = dt$, and $t = \lambda + t_o$, so that

$$\mathcal{F}\left[x(t-t_0)\right] = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\lambda+t_0)}\, d\lambda = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda = e^{-j\omega t_0}\, X(\omega)$$

By following similar steps,

$$\mathcal{F}\left[x(t+t_0)\right] = e^{j\omega t_0}\, X(\omega) \tag{5.16}$$

5.3.4 Frequency Shifting

This property forms a basis for every radio and TV transmitter. This property states that if $X(\omega) = \mathcal{F}\left[x(t)\right]$ and ω_0 is constant, then

$$\mathcal{F}\left[x(t)\, e^{j\omega_0 t}\right] = X(\omega - \omega_0) \tag{5.17}$$

Note that this is a duo of the time-shifting property. The previous equation means that a shifting in the frequency domain is equivalent to a phase shift in the time domain. By definition,

$$\begin{aligned} \mathcal{F}\left[x(t)\, e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} x(t) e^{j\omega_0 t}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t}\, dt = X(\omega - \omega_0) \end{aligned} \tag{5.18}$$

5.3.5 Time Differentiation

If $X(\omega) = \mathcal{F}\left[x(t)\right]$ then the Fourier transform of the derivative of $x(t)$ is given by

$$\mathcal{F}\left[x'(t)\right] = j\omega X(\omega) \tag{5.19}$$

This states that the transform of the derivative of $x(t)$ is obtained by multiplying its transform $X(\omega)$ by $j\omega$. By definition,

$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \tag{5.20}$$

Taking the derivative of both sides with respect to t gives

$$\frac{dx(t)}{dt} = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = j\omega^{\mathcal{F}^{-1}}\left[X(\omega)\right]$$
$$\mathcal{F}\left[x'(t)\right] = j\omega\, X(\omega) \tag{5.21}$$

The nth derivative of eq. (5.19) is

$$\mathcal{F}\left[x^{(n)}(t)\right] = (j\omega)^n\, X(\omega) \tag{5.22}$$

5.3.6 Frequency Differentiation

This property states that if $X(\omega) = \mathcal{F}\left[x(t)\right]$ then

$$\mathcal{F}\left[(-jt)^n\, x(t)\right] = \frac{d^n}{d\omega^n} X(\omega) \tag{5.23}$$

By definition,

$$\frac{d^n}{d\omega^n} X(\omega) = \frac{d^n}{d\omega^n}\left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right) = \int_{-\infty}^{\infty} x(t)\frac{d^n}{d\omega^n} e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{\infty} x(t)(-jt)^n\, e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} (-jt)^n\, x(t)e^{-j\omega t}\,dt \tag{5.24}$$
$$= \mathcal{F}\left((-jt)^n\, x(t)\right)$$

5.3.7 Time Integration

If $X(\omega) = \mathcal{F}\left[x(t)\right]$, then the integration of function $x(t)$ is given by

$$\mathcal{F}\left[\int_{-\infty}^{t} x(t)\,dt\right] = \frac{X(\omega)}{j\omega} + \pi\, X(0)\delta(\omega) \tag{5.25}$$

Integrating $x(t)$ in the time domain leads to division of $X(\omega)$ by $j\omega$ plus an additive term $\pi X(0)\delta(\omega)$ to account for the possibility of the DC component that may appear in the integral of $x(t)$. If we replace ω by 0 in eq. (5.6)

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt \tag{5.26}$$

indicating that the DC component is zero when the integral of $x(t)$ over all time vanishes.

5.3.8 Duality

Duality between the time and frequency domains is another important property of Fourier transforms. This property relates to the fact that the analysis equation and synthesis equation look almost identical except for a factor of $1/2\pi$ and the difference of a minus sign in the exponential in the integral. The duality property states that if $X(\omega)$ is the Fourier transform of $x(t)$, then the Fourier transform of $X(t)$ is $2\pi x(-\omega)$; i.e.,

$$\mathcal{F}\left[x(t)\right] = X(\omega) \Rightarrow \mathcal{F}\left[X(t)\right] = 2\pi x(-\omega) \tag{5.27}$$

This expresses the fact that the Fourier transform pairs are symmetric. To derive the property from eq. (5.7)

$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \tag{5.28}$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\,e^{j\omega t}\,d\omega \tag{5.29}$$

Replacing t with $-t$ in eq. (5.29), we obtain

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\,e^{-j\omega t}\,d\omega \tag{5.30}$$

By interchanging t and ω and comparing the result with eq. (5.28)

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\,e^{-j\omega t}\,dt = \mathcal{F}\left[X(t)\right] \tag{5.31}$$

5.3.9 Convolution

According to the convolution property, the Fourier transform of the convolution of two time functions is the product of their corresponding Fourier transforms. If $x(t)$ and $h(t)$ are two signals, their convolution $y(t)$ is given by the convolution integral

$$y(t) = h(t) * x(t) \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \tag{5.32}$$

If $X(\omega)$, $H(\omega)$, and $Y(\omega)$ are the Fourier transforms of $x(t)$, $h(t)$, and $y(t)$, respectively, then

$$Y(\omega) = \mathcal{F}[h(t) * x(t)] = H(\omega) X(\omega) \tag{5.33}$$

which states that the Fourier transform of the convolution of two time functions is the product of their corresponding Fourier transforms.

To derive the convolution property, we take the Fourier transform on both sides of eq. (5.32); we get

$$Y(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt \tag{5.34}$$

$$Y(\omega) = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} x(t-\tau) e^{-j\omega t}\, dt \right] d\tau$$

By taking $\lambda = t - \tau$ so that $t = \lambda + \tau$ and $dt = d\lambda$:

$$Y(\omega) = \int_{-\infty}^{\infty} h(\tau) \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda+\tau)}\, d\lambda \right] d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda}\, d\lambda = H(\omega) X(\omega) \tag{5.35}$$

Table 5.1 presents these and other properties of the Fourier transform, while Table 5.2 lists the transform pairs of some common functions. Note the similarities between these tables on the Fourier transform and Tables 3.1 and 3.2 on the Laplace transform.

TABLE 5.1
Properties of the Fourier Transform

No.	Property	$x(t)$	$X(\omega)$
1.	Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
2.	Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
3.	Time shift	$x(t-a)$	$e^{-j\omega a}X(\omega)$
4.	Frequency shift	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
5.	Modulation	$\cos(\omega_0 t)x(t)$	$\frac{1}{2}\left[X(\omega+\omega_0)+X(\omega-\omega_0)\right]$
		$\sin(\omega_0 t)x(t)$	$\frac{j}{2}\left[X(\omega+\omega_0)-X(\omega-\omega_0)\right]$
6.	Differentiation in time	$\frac{dx}{dt}$	$j\omega X(\omega)$
		$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
7.	Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
8.	Multiplication by power of t	$t^n x(t)$	$j^n\frac{d^n}{d\omega^n}X(\omega)$
9.	Time integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$
10.	Time reversal	$x(-t)$	$X(-\omega)$ or $X^*(\omega)$
11.	Duality	$X(t)$	$2\pi x(-\omega)$
12.	Time convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
13.	Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
14.	Parseval's relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2 d\omega$

Example 5.4

A signal $x(t)$ has a Fourier transform given by

$$X(\omega)=\frac{5(1+j\omega)}{8-\omega^2+6j\omega}$$

Without finding $f(t)$, find the Fourier transform of the following:

a. $x(t-3)$
b. $x(4t)$
c. $e^{-j2t}x(t)$
d. $x(-2t)$

TABLE 5.2
Fourier Transform Pairs

No.	$x(t)$	$X(\omega)$
1.	$\delta(t)$	1
2.	1	$2\pi\delta(\omega)$
3.	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
4.	$u(t+\tau)-u(t-\tau)$	$2\frac{\sin\omega\tau}{\omega}$
5.	$\lvert t\rvert$	$\frac{-2}{\omega^2}$
6.	$\text{sgn}(t)$	$\frac{2}{j\omega}$
7.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
8.	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
9.	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$
10.	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
11.	$\lvert t\rvert e^{-at}\,u(t)$	$\frac{4aj\omega}{a^2+\omega^2}$
12.	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
13.	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
14.	$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
15.	$(\sin\omega_0 t)u(t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$
16.	$(\cos\omega_0 t)u(t)$	$\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
17.	$\sin(\omega_0 t+\theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0)-e^{j\theta}\delta(\omega-\omega_0)]$
18.	$\cos(\omega_0 t+\theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0)+e^{j\theta}\delta(\omega-\omega_0)]$
19.	$e^{-at}\sin\omega_o t u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
20.	$e^{-at}\cos\omega_o t u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

(*Continued*)

TABLE 5.2 (*Continued*)
Fourier Transform Pairs

No.	$x(t)$	$X(\omega)$
21.	$\Pi(t/\tau)$	$\tau \sin c\left(\frac{\omega\tau}{2}\right)$
22.	$\Lambda\left(\frac{t}{\tau}\right)$	$\tau \sin c^2\left(\frac{\omega\tau}{2}\right)$
23.	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$

SOLUTION

We apply the relevant property for each case.

a. $\mathcal{F}[x(t-3)] = e^{-j\omega 3}X(\omega) = \dfrac{5(1+j\omega)e^{-j\omega 3}}{8-\omega^2+j6\omega}$

b. $\mathcal{F}[x(4t)] = \dfrac{1}{4}X\left(\dfrac{\omega}{4}\right) = \dfrac{\frac{5}{4}(1+j\omega/4)}{8-\omega^2/16+j6\omega/4} = \dfrac{5(4+j\omega)}{128-\omega^2+j24\omega}$

c. $\mathcal{F}[e^{-j2t}x(t)] = X(\omega+2) = \dfrac{5[1+j(\omega+2)]}{8-(\omega+2)^2+6j(\omega+2)} = \dfrac{5(1+j\omega+j2)}{4-\omega^2-4\omega+6j\omega+j12}$

d. $\mathcal{F}[x(-2t)] = \dfrac{1}{2}X\left(\dfrac{\omega}{-2}\right) = \dfrac{\frac{5}{2}(1-j\omega/2)}{8-\frac{\omega^2}{4}-\frac{6j\omega}{2}} = \dfrac{5(2-j\omega)}{32-\omega^2-12j\omega}$

Practice Problem 5.4 A signal $x(t)$ has the Fourier transform

$$X(\omega) = \frac{9}{9+\omega^2}$$

Find the Fourier transform of the following signals:

a. $y(t) = x(2t-1)$
b. $z(t) = dx(2t)/dt$
c. $h(t) = \int_{-\infty}^{t} x(\lambda)d\lambda$

Answer: (a) $\dfrac{18e^{-j\omega/2}}{36+\omega^2}$, (b) $\dfrac{18j\omega}{36+\omega^2}$, (c) $\pi\delta(\omega)+\dfrac{9}{j\omega(9+\omega^2)}$.

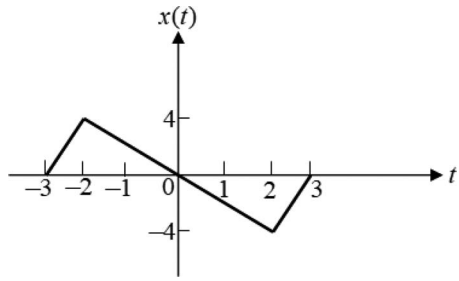


FIGURE 5.6 For Example 5.5.

Example 5.5

Determine the Fourier transform of the signal in Figure 5.6.

SOLUTION

Although the Fourier transform of $x(t)$ can be found directly using eq. (5.6), it is much easier to find it using the derivative property. Taking the first derivative of $g(t)$ produces the signal in Figure 5.7(a). Taking the second derivatives gives us the signal in Figure 5.7(b). From this,

$$x''(t) = 4\delta(t+3) - 6\delta(t+2) + 6\delta(t-2) - 4\delta(t-3)$$

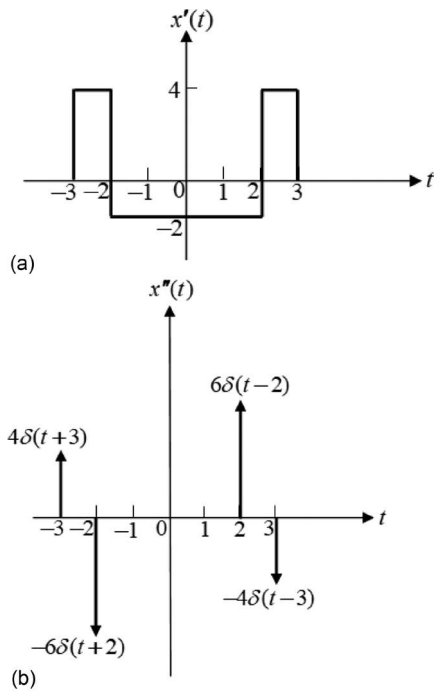


FIGURE 5.7 For Example 5.5: (a) first derivative of $x(t)$; (b) second derivative of $x(t)$.

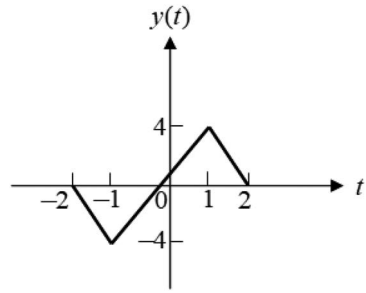


FIGURE 5.8 For Practice Problem 5.5.

Taking the Fourier transform of each term, we obtain

$$(j\omega)^2 X(\omega) = 4e^{j3\omega} - 6e^{j2\omega} + 6e^{-j2\omega} - 4e^{-j3\omega}$$
$$-\omega^2 X(\omega) = 4(e^{j3\omega} - e^{-j3\omega}) + 6(e^{j2\omega} - e^{-j2\omega})$$
$$= j8\sin 3\omega - j12\sin 2\omega$$
$$X(\omega) = \frac{j}{\omega^2}(12\sin 2\omega - 8\sin 3\omega)$$

Practice Problem 5.5 Determine the Fourier transform of the function in Figure 5.8.

Answer: $Y(\omega) = \dfrac{j10}{\omega^2}(\sin 2\omega - 2\sin\omega)$.

5.4 INVERSE FOURIER TRANSFORM

In general, we would use eq. (5.7) to find the inverse Fourier transform $x(t)$ of $X(\omega)$. Quite often, the Fourier transform is in the form of a rational function. In this case, we perform a partial fraction expansion of $X(\omega)$ and find the corresponding time-domain signals using the Fourier transform pairs in Table 5.2. The procedure is similar to what we did in Section 3.4 for the Laplace transform.

Example 5.6

Find the inverse Fourier transform of:

a. $G(\omega) = \dfrac{10\,j\omega}{(-j\omega + 2)(j\omega + 3)}$

b. $Y(\omega) = \dfrac{\delta(\omega)}{(j\omega + 1)(j\omega + 2)}$

SOLUTION

a. To avoid complex algebra, let $s = j\omega$. Using a partial fraction,

$$G(s) = \frac{10s}{(2-s)(3+s)} = \frac{-10s}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}, \quad s = j\omega$$

$$A = (s-2)G(s)\big|_{s=2} = \frac{-10(2)}{2+3} = -4$$

$$B = (s+3)G(s)\big|_{s=-3} = \frac{-10(-3)}{-3-2} = -6$$

$$G(\omega) = \frac{-4}{j\omega - 2} - \frac{6}{j\omega + 3}$$

Taking the inverse Fourier transform of each term,

$$g(t) = -4e^{2t}u(-t) - 6e^{-3t}u(t)$$

b. Because of the delta function, we use eq. (5.7) to find the inverse.

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\delta(\omega)e^{j\omega t}d\omega}{(2+j\omega)(j\omega+1)} = \frac{1}{2\pi}\frac{e^{j\omega t}}{(2+j\omega)(j\omega+1)}\bigg|_{\omega=0} = \frac{1}{2\pi}\frac{1}{2} = \frac{1}{4\pi}$$

where the shifting property has been applied.

Practice Problem 5.6 Obtain the inverse Fourier transform of:

a. $F(\omega) = \dfrac{e^{-j2\omega}}{1+j\omega}$

b. $G(\omega) = \dfrac{\pi\delta(\omega)}{(5+j\omega)(2+j\omega)}$

Answer: (a) $f(t) = e^{-(t-2)}u(t-2)$, (b) $g(t) = 0.05$.

5.5 APPLICATIONS

The Fourier transform is used extensively in a variety of fields such optics, spectroscopy, acoustics, computer science, and electrical engineering. In electrical engineering, it is applied in circuit analysis, communications systems, and signal processing. Here we consider three application areas: circuit analysis, amplitude modulation (AM),

and sampling. Filtering is another area of application, but we already considered this in Section 4.5.3.

5.5.1 Circuit Analysis

We use Fourier transform to analyze circuits with nonsinusoidal excitations in exactly the same way we use phasor techniques to analyze circuits with sinusoidal excitations. For example, Ohm's law is written as

$$V(\omega) = Z(\omega)\, I(\omega) \tag{5.36}$$

where $V(\omega)$ and $I(\omega)$ are the Fourier transforms of the voltage and current, respectively, and $Z(\omega)$ is the impedance. The impedances of resistor, inductors, and capacitors are the same as in phasor analysis, i.e.,

$$\begin{aligned} R &\Rightarrow R \\ L &\Rightarrow j\omega L \\ C &\Rightarrow \frac{1}{j\omega C} \end{aligned} \tag{5.37}$$

The Fourier approach to circuit analysis generalizes the phasor technique. It involves three steps. First, we transform the circuit elements into the frequency domain as in eq. (5.56) and take the Fourier transform of the excitations. Second, we apply circuit techniques such as Kirchhoff's voltage law, Kirchhoff's current law, voltage division, current division, source transformation, or node or mesh analysis to find the unknown response (current or voltage). Third, we finally take the inverse Fourier transform to get the response in the time domain. (We should note that Fourier analysis cannot handle circuits with initial conditions.)

As usual, the transfer function $H(\omega)$ is the ratio of the output response $Y(\omega)$ to the input excitation $X(\omega)$:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{5.38}$$

or

$$Y(\omega) = H(\omega)X(\omega) \tag{5.39}$$

Eq. (5.58) states that if we know the transfer function $H(\omega)$ and the input $X(\omega)$, we can find the output $Y(\omega)$.

Example 5.7

Determine $v_o(t)$ in the circuit of Figure 5.9, where $i_s = 10e^{-2t}u(t)$A.

SOLUTION

$$0.5\text{F} \rightarrow \frac{1}{j\omega C} = \frac{1}{j\omega 0.5} = \frac{2}{j\omega},$$

$$i_s = 10e^{-2t} \rightarrow I_s = \frac{10}{2 + j\omega}$$

Using current division, the current through the capacitor is

$$I_o = \frac{2}{2 + \frac{2}{j\omega}} I_s = \frac{j\omega}{1 + j\omega} \cdot \frac{10}{2 + j\omega}$$

$$V_o = I_o \frac{2}{j\omega} = \frac{20}{(1 + j\omega)(2 + j\omega)} = \frac{20}{(s+1)(s+2)}, \quad s = j\omega$$

$$V_o = \frac{A}{s+1} + \frac{B}{s+2} = 20\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

Taking the inverse Fourier transform produces

$$v_o(t) = 20(e^{-t} - e^{-2t})u(t)\ \text{V}$$

Practice Problem 5.7 Find $i_o(t)$ in the circuit of Figure 5.10. Let $v_s(t) = 4e^{-2t}u(t)$.

Answer: $i(t) = 4(e^{-0.5t} - e^{-t})u(t)$A.

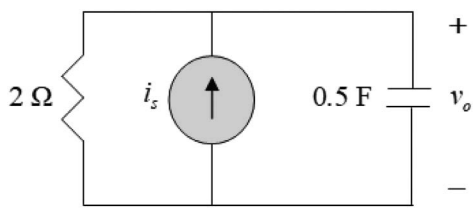


FIGURE 5.9 For Example 5.7.

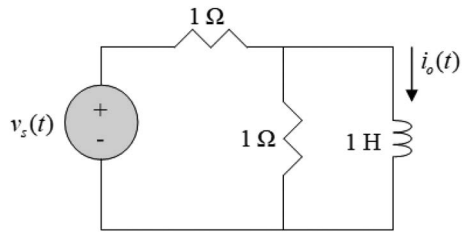


FIGURE 5.10 For Practice Problem 5.7.

Example 5.8

Given the circuit in Figure 5.11, with its excitation, determine the Fourier transform of $i(t)$.

SOLUTION

One easy way to find the Fourier transform of v_s is to first find its derivative.

$$v_s'(t) = 8\delta(t) - 8\delta(t-1)$$

Taking the Fourier transform of each side,

$$j\omega V_s = 8(1 - e^{-j\omega}) \rightarrow V_s = \frac{8}{j\omega}(1 - e^{-j\omega})$$

The current $I(\omega)$ is obtained as

$$I(\omega) = \frac{V_s}{3 + j\omega 1} = \frac{8(1 - e^{-j\omega})}{j\omega(3 + j\omega)}$$

Let $X(\omega) = \dfrac{8}{j\omega(3 + j\omega)}$, so that $I(\omega) = X(\omega)(1 - e^{-j\omega})$

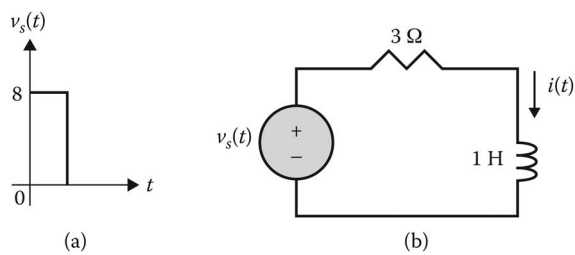


FIGURE 5.11 For Example 5.8.

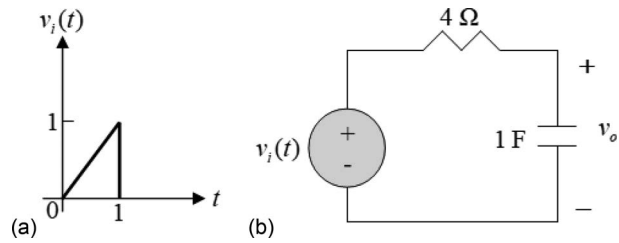


FIGURE 5.12 For Practice Problem 5.8.

$$X(s) = \frac{8}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}, \quad s = j\omega$$

$$A = sX(s)\big|_{s=0} = \frac{8}{3}$$

$$B = (s+3)X(s)\big|_{s=-3} = -\frac{8}{3}$$

$$X(\omega) = \frac{8}{3}\left(\frac{1}{j\omega} - \frac{1}{3+j\omega}\right)$$

$$x(t) = \frac{8}{3}\left(\frac{1}{2}\text{sgn}(t) - e^{-3t}u(t)\right)$$

Since, $I(\omega) = X(\omega)(1 - e^{-j\omega})$,

$$\begin{aligned} i(t) &= x(t) - x(t-1) \\ &= \frac{4}{3}\text{sgn}(t) - \frac{8}{3}e^{-3t}u(t) - \frac{4}{3}\text{sgn}(t-1) - \frac{8}{3}e^{-3(t-1)}u(t-1) \end{aligned}$$

Practice Problem 5.8 Determine the voltage $V_o(\omega)$ in the circuit of Figure 5.12(b), given the voltage source shown in Figure 5.12(a).

Answer: $V_o(\omega) = \dfrac{1}{\omega^2(1+4j\omega)}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$.

5.5.2 Amplitude Modulation

AM is one application of the Fourier transform. Transmission of information through space has become part of a modern society, especially with wireless telephony. However, transmitting intelligent signals—such as for speech and music—is not practical: it requires a lot of power and large antennas. One way of transmitting low-frequency audio information (50 Hz to 20 kHz) is to transmit it along with a high-frequency signal, called a *carrier*. Any of the three characteristics (amplitude, frequency, or phase) of a carrier can be controlled to be able to carry the intelligent

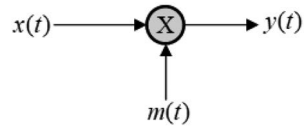


FIGURE 5.13 Modulated signal.

signal, called the *modulating signal*. Here we are interested in the control of the carrier's amplitude. This is known as *amplitude modulation*.

Amplitude modulation (AM) is a process whereby we let the modulating signal control the amplitude of the carrier.

Consider the signal multiplier shown in Figure 5.13. We focus only on the case when the modulating signal $m(t) = \cos \omega_o t$. The output of the multiplier is

$$y(t) = x(t) \cos \omega_o t \tag{5.40}$$

Since the multiplication in the time domain always results in convolution in the frequency domain,

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} X(\omega) * \pi \left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o) \right] \\ &= \frac{1}{2} \left[X(\omega - \omega_o) + X(\omega + \omega_o) \right] \end{aligned} \tag{5.41}$$

Thus, modulation results in shifting the spectrum of the signal as illustrated in Figure 5.14. The signal with frequency $\omega_c - \omega_o$ is known as the *lower sideband*, while the one with frequency $\omega_c + \omega_o$ is known as the *upper sideband*.

At the receiving end of the transmission, the audio information is received from the modulated carrier by a process known as *demodulation*. This shifts back the message spectrum to its original low-frequency location. A synchronous demodulation multiplies the modulated signal $y(t)$ with $\cos \omega_o t$ as shown in Figure 5.15.

$$z(t) = y(t) \cos \omega_o t \tag{5.42}$$

Taking the Fourier transform as we did before,

$$\begin{aligned} Z(\omega) &= \frac{1}{2} \left[Y(\omega - \omega_o) + Y(\omega + \omega_o) \right] \\ &= \frac{1}{2} X(\omega) + \frac{1}{4} X(\omega - 2\omega_o) + \frac{1}{4} X(\omega + 2\omega_o) \end{aligned} \tag{5.43}$$

This is illustrated in Figure 5.16. The signal $z(t)$ is passed through a lowpass filter [see Figure 5.17(a)], which passes only the low-frequency components. Thus $x(t)$ is extracted as shown in Figure 5.17(b).

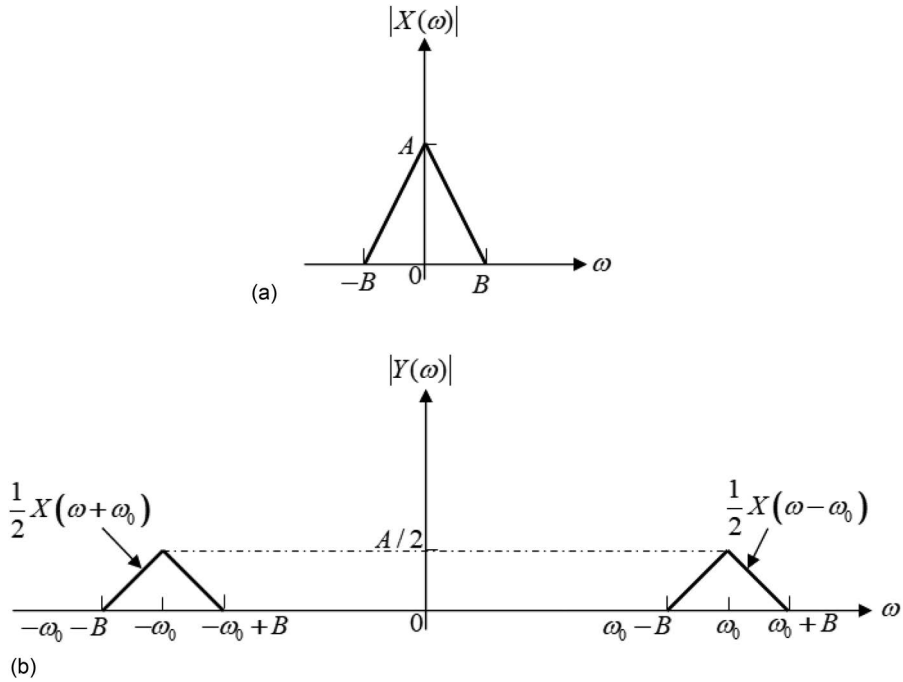


FIGURE 5.14 Amplitude spectra of (a) signal $x(t)$ and (b) modulated signal $y(t)$.

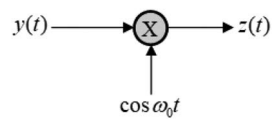


FIGURE 5.15 Demodulated signal.

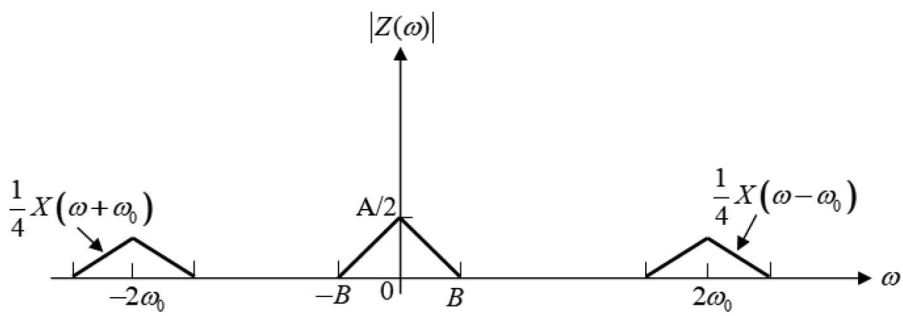


FIGURE 5.16 Amplitude spectra of $z(t)$.

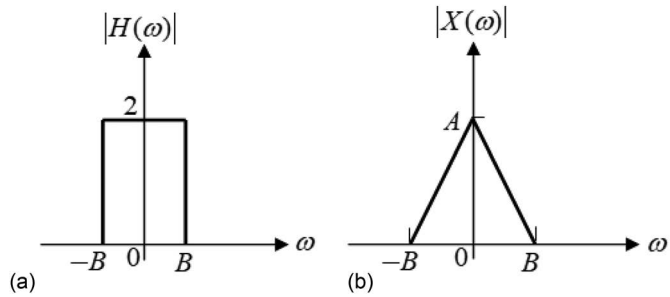


FIGURE 5.17 (a) Low-pass filter; (b) spectrum of the original signal.

Example 5.9

An AM signal is given by

$$x(t) = \cos 200\pi t \cos 10^4 t$$

Identify the upper sideband and the lower sideband.

SOLUTION

$$\omega_o = 200\pi \rightarrow f_o = \frac{\omega_o}{2\pi} = 100$$

$$\omega_c = 10^4\pi \rightarrow f_c = \frac{\omega_c}{2\pi} = 5000$$

$$\text{USB} = f_c + f_o = 5{,}100 = 5.1\,\text{kHz}$$

$$\text{LSB} = f_c - f_o = 4{,}900 = 4.9\ \text{kHz}$$

Practice Problem 5.9 A music signal whose frequencies range from 80 Hz to 12 kHz is used to modulate a 2-MHz carrier. Find the range of frequencies for the lower and upper sidebands.

Answer: Lower sideband: 1,999,200 Hz to 1,988,000 Hz; upper sideband: 2,000,080 Hz to 2,012,000 Hz.

5.5.3 Sampling

Sampling is an important operation in signal processing. It may be regarded as a way of reducing analog signals to discrete signals. Thus, sampling is the bridge from

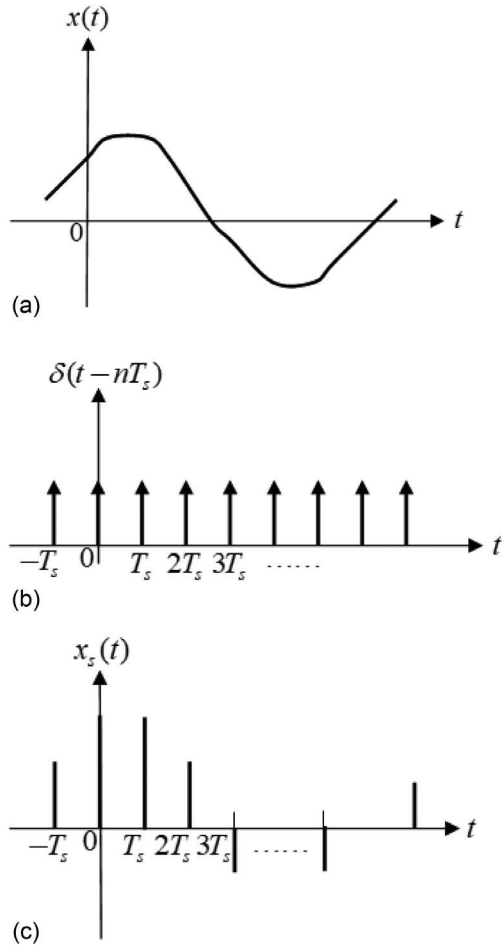


FIGURE 5.18 (a) Continuous (analog) signal to be sampled, (b) train of impulses, and (c) sampled (digital) signal.

continuous to discrete signals. In analog systems, we process the entire signals, but in digital systems, only samples of signals are necessary. The sampling can be done by using a train of pulses or impulses. We will use impulse sampling here.

Consider the continuous-time signal $x(t)$ depicted in Figure 5.18(a). We multiply by a train of impulses $\delta(t - nT_s)$ as shown in Figure 5.18(b), where T_s is the *sampling interval* and $f_s = 1/T_s$ is the *sampling frequency* (or *rate)*. The value of $x(t)$ at point nT_s is

$$x(t)\delta(t - nT_s) = x(nT_s)\delta(t - nT_s) \tag{5.44}$$

The sampled signal $x_s(t)$ in Figure 5.18(c) is therefore

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) \tag{5.45}$$

We take the Fourier transform of this and apply the frequency convolution property.

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi}\left[X(\omega)\right] * \omega_s \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_s) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \end{aligned} \tag{5.46}$$

where $\omega_s = 2\pi/T_s$. We have used item 23 in Table 5.2. Equation (5.46) states that the Fourier transform $X_s(\omega)$ of the sampled signal is a sum of translates of the Fourier transform of the original signal $X(\omega)$ at a rate of $1/T_s$.

To ensure that the sampling process is successful, we must carefully select the sampling interval. This is stated in the following sampling theorem:

> A band-limited signal, with bandwidth W hertz, may be completely recovered from its samples if taken at a frequency at least twice as high as $2W$ samples per second.

(In practice, a signal is made band-limited by passing it through a lowpass filter before sampling it.) For a signal with no frequency component higher than W hertz, there is no loss of information if the sampling frequency is at least twice the highest frequency in the signal, i.e.,

$$\frac{1}{T_s} = f_s \geq 2W \tag{5.47}$$

The minimum sampling frequency $f_s = 2W$ is known as the *Nyquist frequency* or rate, and the maximum spacing $1/f_s$ is the *Nyquist interval*, named after Harry Nyquist (1889–1976) for his paper published in 1928. The Nyquist rate is the minimum sampling rate necessary to preserve the information $x(t)$. A signal is said to be oversampled if it is sampled at a rate greater than its Nyquist rate. It is undersampled if it is sampled at less than the Nyquist rate.

Example 5.10

For frequencies above 4 kHz, the spectrum of a signal is zero. To sample the signal, find the maximum time spacing between samples.

SOLUTION

$$W = 4 \text{ kHz}$$

The Nyquist rate is

$$f_s = 2W = 8 \text{ kHz}$$

The Nyquist interval

$$T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3} = 125 \, \mu\text{s}$$

Practice Problem 5.10 A voice signal is band-limited to 2.5 kHz. Find the Nyquist rate and the Nyquist interval for the signal.

Answer: 5 kHz, 0.2 milliseconds.

5.6 PARSEVAL'S THEOREM

Parseval's theorem demonstrates another application of the Fourier transform. It states the relationship between energy in the time and frequency domains. The theorem is important in communications and signal processing.

If $p(t)$ is the power associated with the signal, the energy carried by the signal is

$$E = \int_{-\infty}^{\infty} p(t)\, dt \tag{5.48}$$

It is convenient to use a 1-Ω resistor as the base for energy calculation. For a 1-Ω resistor, $p(t) = v^2(t) = i^2(t) = x^2(t)$, where $x(t)$ stands for either voltage or current. Thus eq. (5.48) becomes

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)\, dt \tag{5.49}$$

Parseval's theorem states that this same energy can be calculated in the frequency domain as

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \tag{5.50}$$

indicating that the Fourier transform is an energy-conserving relation.

Parseval's theorem states that the total energy delivered to a 1-Ω resistor equals the total area under the square $x^2(t)$ or $1/2\pi$ times the total area under the magnitude of the Fourier transform squared $|X(\omega)|^2$.

Parseval's theorem is named after the French mathematician Marc-Antoine Parseval des Charles (1755–1836). It relates energy between the time domain and the frequency domain. It provides the physical significance of $X(\omega)$, namely, that $|X(\omega)|^2$ is the energy density (in joules per hertz) corresponding to $x(t)$. The plot of $|X(\omega)|^2$ versus ω is the *energy spectrum* of the signal $x(t)$.

To derive eq. (5.50), we begin with eq. (5.49) and substitute eq. (5.7) for one of the $x(t)$'s.

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right]dt \tag{5.51}$$

The function $x(t)$ can be moved inside the inner integral, since the integral does not involve time

$$E_{1\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)X(\omega)e^{j\omega t}d\omega dt \tag{5.52}$$

Reversing the order of integration:

$$\begin{aligned} E_{1\Omega} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X(-\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega \end{aligned} \tag{5.53}$$

But $X(\omega)X^*(\omega) = |X(\omega)|^2$. Hence,

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \tag{5.54}$$

as expected. Eq. (5.54) shows the energy of $x(t)$ can be computed directly in the time domain or indirectly from $X(\omega)$ in the frequency domain.

Since $|X(\omega)|^2$ is an even function, we may integrate from 0 to ∞ and double the result; that is,

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|^2 \, d\omega \tag{5.55}$$

We may calculate the energy in any frequency band $\omega_1 < \omega < \omega_2$ as

$$E_{1\Omega} = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2 \, d\omega \tag{5.56}$$

We should note that Parseval's theorem as stated here applies to nonperiodic signals. Parseval's theorem for periodic signals was covered in Section 4.4.5

Example 5.11

A bandpass filter has its lower and upper cutoff frequencies as 10 Hz and 20 Hz, respectively. If the input signal is $V_i = 4e^{-t}u(t)$, calculate the 1-Ω energy of the input and the percentage that appears at the output.

SOLUTION

The energy of the input signal is

$$E_i = \int_{-\infty}^{\infty} v_i^2(t)dt = \int_{0}^{\infty} 16e^{-2t}dt = \frac{16}{-2}e^{-2t}\Big|_0^{\infty} = 8 \text{ J}$$

The Fourier transform of the input signal is

$$V_i = \frac{4}{1 + j\omega}$$

The energy in the interval $10 < f < 20$ or $20\pi < \omega < 40\pi$ is

$$E_o = \frac{1}{2\pi}\int_{20\pi}^{40\pi} |V_i(\omega)|^2 \, d\omega = \frac{1}{2\pi}\int_{20\pi}^{40\pi} \frac{16}{(1+\omega^2)}d\omega = \frac{8}{\pi}\int_{20\pi}^{40\pi} \frac{d\omega}{1+\omega^2}$$

But $\int \frac{dx}{a^2+\omega^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$E_o = \frac{8}{\pi}\tan^{-1}\omega\Big|_{20\pi}^{40\pi} = \frac{8}{\pi}(1.5628 - 1.5549) = 0.02012$$

$$\frac{E_o}{E_i} \times 100 = \frac{2.012}{8} = 0.252\%$$

Practice Problem 5.11 If $x(t) \Leftrightarrow X(\omega)$ and

$$X(\omega) = \begin{cases} 10, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$

Let $y(t) = \frac{d^2x(t)}{dt^2}$. Find $\int_{-\infty}^{\infty} |y(t)|^2 \, dt$

Answer: 2010.62 J.

5.7 COMPARING THE FOURIER AND LAPLACE TRANSFORMS

The Fourier and Laplace transforms are closely related, as evident in their definitions, which are repeated here for convenience. The general (bilateral) Laplace transform of $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \tag{5.57}$$

where $s = \sigma + j\omega$. If we let $s = j\omega$, eq. (5.57) becomes

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \tag{5.58}$$

which is the Fourier transform of $x(t)$. For a causal signal $x(t)$ [i.e., positive-time signal, $x(t) = 0$, $t < 0$] and $\int_{-\infty}^{\infty} |x(t)| dt < \infty$, the two transforms are related by

$$X(\omega) = X(s)\big|_{s=j\omega} \tag{5.59}$$

showing that the Fourier transform is a special case of the Laplace transform with $s = j\omega$. We can use eq. (5.59) to obtain the Fourier transform of signals for which the

Fourier integral converges. This integral will converge when all the poles of $F(s)$ lie in the left half of the s-plane and not on the $j\omega$ axis.

The Laplace transform defined and used in Chapter 4 is one-sided (or unilateral) in that the integral is over $0 < t < \infty$, restricting its usefulness to only positive-time signals, $x(t)$, $t > 0$. The Fourier transform accommodates signals existing for all time.

Although the Laplace transform has some advantages over the Fourier transform in circuit analysis, being an extension of Fourier analysis to nonperiodic signals, the Fourier transform uses the same frequency-domain representation as phasor analysis. The Fourier transform shares some operational properties with the Laplace transform but has some unique properties as well. For circuit analysis, the Laplace transform is more widely used because the Laplace integral converges for a wider range of signals and automatically incorporates initial conditions.

5.8 COMPUTING WITH MATLAB®

The MATLAB function **fourier** can be used to find the Fourier transform of a signal $x(t)$, while the function **ifourier** can be used to find the inverse Fourier transform of $X(\omega)$. How to use them can be displayed with the **help Fourier** and **help ifourier** commands. Since transfer functions are already functions of frequency, it is straightforward and easy to plot them using MATLAB.

Example 5.12

Use MATLAB to find the Fourier transform of

$$x(t) = u(t-2) - e^{-2t}u(t)$$

SOLUTION

The MATLAB script for finding the Fourier transform of $x(t)$ is as follows:

```
syms  x t
 x=heaviside(t-2)-exp(-2*t)*Heaviside(t) %define the signal
 X = fourier(x)
pretty(X)  %  simplify
```

This returns

```
x =
heaviside(t - 2) - heaviside(t)/exp(2*t)
X =
 - (1/exp(2*w*i))*(- pi*dirac(-w) + i/w) - 1/(2 + w*i)

          1     /                   i \       1
 - ---------- | - pi dirac(-w) + -   | - -------
    exp(2 w i) \                   w /   2 + w i
```

From this, we obtain

$$X(\omega) = e^{-j2\omega}\left[-\pi\delta(-\omega) + \frac{j}{\omega}\right] - \frac{1}{2 + j\omega}$$

Practice Problem 5.12 Let $y(t) = 4\delta(t) + 2e^{-t}\,u(t)$. Use MATLAB to find $Y(\omega)$.

Answer: $Y(\omega) = \dfrac{2}{1 + j\omega} + 4$.

Example 5.13

Use MATLAB to find the inverse Fourier transform of

$$X(\omega) = \frac{2j\omega}{1 + j\omega}$$

SOLUTION

The following MATLAB script is used to find the inverse of $X(\omega)$.

```
syms  X  x  w
X = 2*j*w/( 1 + j*w);
x = ifourier(X)
pretty(x)
```

This returns

```
x =
((4*pi*dirac(x))/exp(x) - (4*pi*heaviside(x))/exp(x))/(2*pi)
 4 PI exp(-x) dirac(x) - 4 PI exp(-x) heaviside(x)
 -------------------------------------------------
                        2 PI
```

From this, we get

$$x(t) = 2e^{-t}\delta(t) - 2e^{-t}u(t)$$

Practice Problem 5.13 Use MATLAB to find the inverse Fourier transform of

$$X(\omega) = \frac{\delta(\omega)}{1 + j2\omega}$$

Answer: $1/(2\pi)$.

Example 5.14

Consider a system with transfer function

$$H(s) = \frac{100s^2}{s^4 + 25s^3 + 50s^2 + 400s + 6000}, \quad s = j\omega$$

Use MATLAB to obtain the plot of $H(\omega)$.

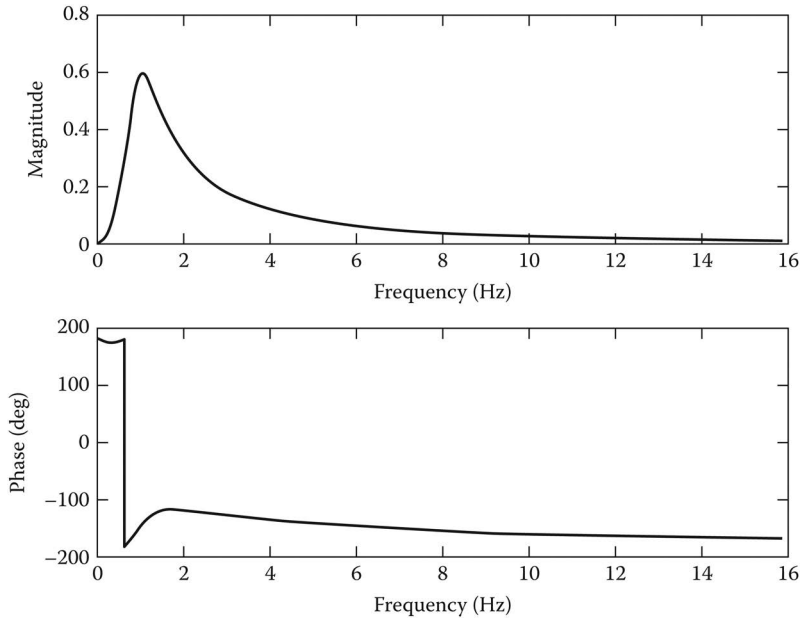


FIGURE 5.19 For Example 5.14.

SOLUTION

The MATLAB script is as follows. Running the script results in the plot shown in Figure 5.19.

```
num = [100 0 0];
den = [ 1  25  50  400  6000];
w = 0.1:0.1:100
H = freqs(num, den,w);
mag =abs(H);
phase = angle(H)*180/pi   % converts phase to degrees
subplot(2,1,1)
 plot(w/(2*pi), mag)
  xlabel('Frequency (Hz)')
  ylabel('Magnitude')
subplot(2,1,2)
 plot(w/(2*pi), phase)
  xlabel('Frequency (Hz)')
ylabel('Phase (deg)')
```

Practice Problem 5.14 Use MATLAB to plot the frequency response to the transfer function

$$H(s) = \frac{1}{s^3 + 7s^2 + 14s + 8}$$

Answer: See Figure 5.20.

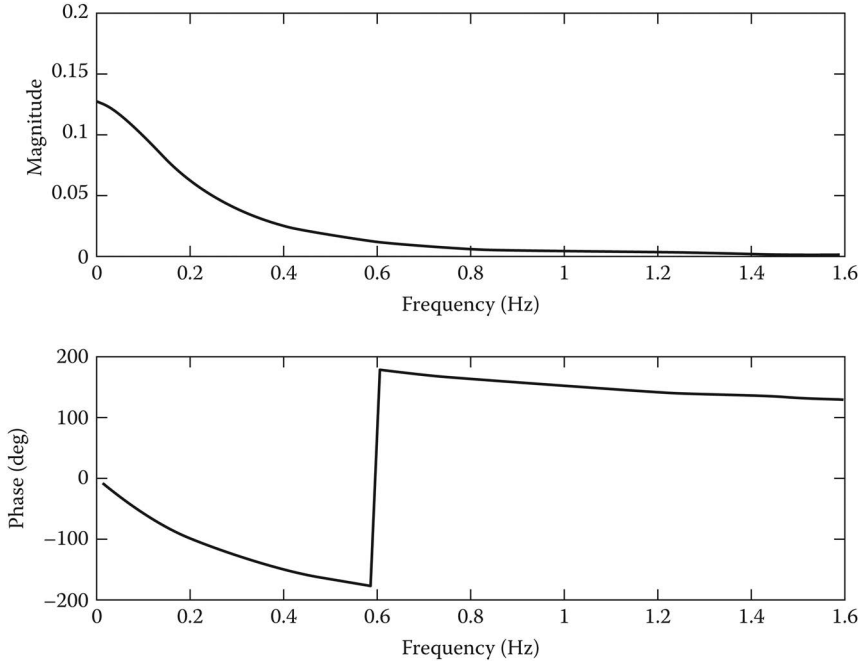


FIGURE 5.20 For Practice Problem 5.14.

SUMMARY

1. The Fourier transform is defined as

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

   It converts a nonperiodic signal $x(t)$ into a transform $X(\omega)$.
2. The inverse Fourier transform of $X(\omega)$ is defined as

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

3. The magnitude $|X(\omega)|$ plotted against ω is known as the magnitude spectrum of $x(t)$. while the plot of $|X(\omega)|^2$ against ω is known as the energy spectrum. The angle of $X(\omega)$ plotted against ω is known as the phase spectrum.
4. Common Fourier transform properties and pairs are in Tables 5.1 and 5.2, respectively.
5. If $H(\omega)$ is the transfer function of a system, the output $Y(\omega)$ can be obtained from the input $X(\omega)$ using

$$Y(\omega) = H(\omega)X(\omega)$$

6. Parseval's theorem provides the energy relationship between a signal $x(t)$ and its Fourier transform $X(\omega)$. The 1-Ω energy is

$$E_{1\Omega} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

7. Applications of the Fourier transform include circuit analysis, AM, and sampling. Applying the Fourier transform approach to circuit analysis involves transforming the excitation and circuit elements into the frequency domain, solving for the unknown response, and transforming the response to the time domain using the inverse Fourier transform. For AM, the modulating signal is used to control the amplitude of the carrier. For sampling application, we found that no information is lost in sampling if the sampling frequency is equal to at least twice the Nyquist rate.
8. If $x(t) = 0$ for $t \le 0$, we can obtain the Fourier transform of signal $x(t)$ from the one-sided Laplace transform $X(s)$ of $x(t)$ by replacing s with $j\omega$.
9. MATLAB commands can be used to find the Fourier transform of a signal or its inverse.

REVIEW QUESTIONS

5.1 The Fourier transform of $x(t)$ is called the spectrum of $x(t)$.
(a) True, (b) False

5.2 The Fourier transform and the Laplace transform of $\delta(t)$ are the same.
(a) True, (b) False.

5.3 Which of these functions is not Fourier transformable?
(a) $e^t u(-t)$, (b) $te^{-3t}u(t)$, (c) $1/t$, (d) $|t|u(t)$

5.4 If $x(t) = 2\delta(t)$ and $y(t) = dx/dt$, then $Y(\omega)$ is:
(a) $2\delta'(\omega)$, (b) $2\delta(\omega)$, (c) $2j\omega$, (d) $2/j\omega$

5.5 The inverse Fourier transform of $\dfrac{e^{-j\omega}}{3+j\omega}$ is
(a) e^{-3t}, (b) $e^{-3t}u(t-1)$, (c) $e^{-3(t-1)}$, (d) $e^{-3(t-1)}u(t-1)$

5.6 The property that relates $e^{j\omega_0 t}X(t) \Leftrightarrow X(\omega - \omega_o)$ is:
(a) time shift, (b) frequency shift, (c) time differentiation, (d) frequency differentiation.

5.7 Evaluating the integral $\displaystyle\int_{-\infty}^{\infty} \frac{8\delta(\omega-2)}{4+\omega^2}d\omega$ gives:
(a) 0, (b) 1, (c) 2, (d) ∞

5.8 A unit step current is applied through a 1-H inductor with no initial current. The voltage across the inductor is:
(a) $u(t)$, (b) $\text{sgn}(t)$, (c) $e^{-t}u(t)$, (d) $\delta(t)$

5.9 The total energy of a signal is independent of its phase spectrum
(a) True, (b) False

5.10 The MATLAB command for finding the inverse Fourier transform is:
(a) invfourier, (b) ifourier, (c) fourierinv, (d) fourier

Answer: 5.1 a, 5.2 a, 5.3 c, 5.4 c, 5.5 d, 5.6 b, 5.7 b, 5.8 d, 5.9 b, 5.10 b.

PROBLEMS

Sections 5.2 and 5.3: Fourier Transform and its Properties

5.1 Determine the Fourier transform of a pulse signal defined as

$$x(t) = \begin{cases} 5, & -2 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

Simplify your work.

5.2 Obtain the Fourier transforms of the following signals:

(a) $x_1(t) = \begin{cases} 1, & -2 < t < -1 \\ 0, & -1 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $x_2(t) = \begin{cases} -1, & -2 < t < -1 \\ 0, & -1 < t < 1 \\ 1, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$

5.3 Find the Fourier transform of $z(t)$ in Figure 5.21.

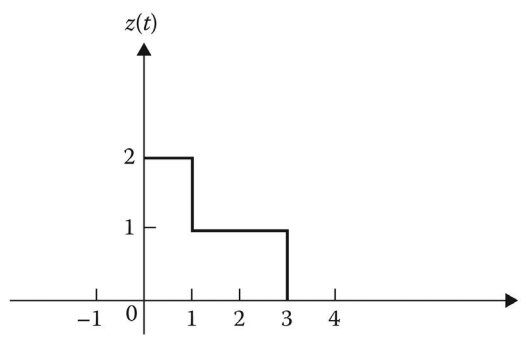


FIGURE 5.21 For Problem 5.3.

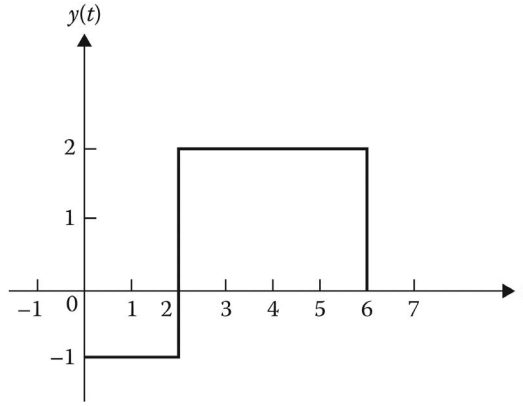


FIGURE 5.22 For Problem 5.4.

5.4 Obtain the Fourier transform of the signal $y(t)$ shown in Figure 5.22.
5.5 Determine the Fourier transform of the signal $x(t)$ in Figure 5.23.
5.6 Find the Fourier transform of the signal $z(t)$ in Figure 5.24.
5.7 Calculate the Fourier transforms of the signals in Figure 5.25.
5.8 Compute the Fourier transform of the signal shown in Figure 5.26.

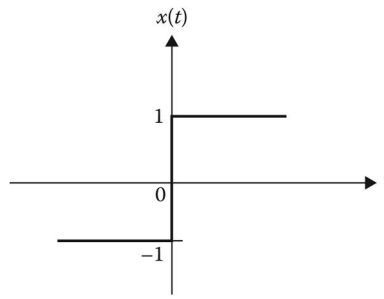


FIGURE 5.23 For Problem 5.5.

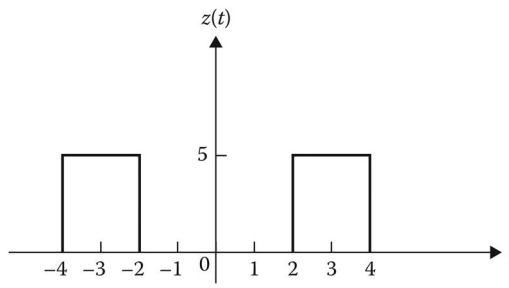


FIGURE 5.24 For Problem 5.6.

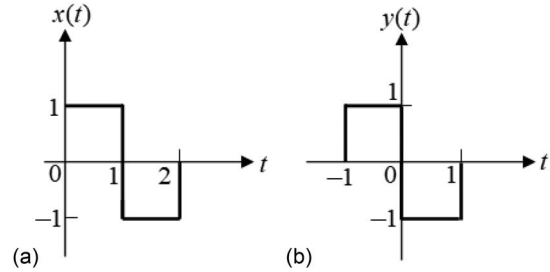


FIGURE 5.25 For Problem 5.7.

5.9 Find the Fourier transform of:
(a) $x(t) = \delta(t) - \delta(t-2)$
(b) $y(t) = u(t+1) - u(t-1)$

5.10 Using the properties of the Fourier transform, determine the Fourier transform of these signals.
(a) $x(t) = \sin(2t)u(t)$
(b) $y(t) = e^{-t}\cos(3t)u(t)$
(c) $z(t) = te^{-2t}u(t)$

5.11 Verify that:
(a) $\int_{-\infty}^{\infty} \cos\omega t d\omega = 2\pi\delta(t)$

(b) $\frac{j}{2}\int_{-\infty}^{\infty}\left[\delta(\omega+\omega_o)-\delta(\omega-\omega_o)\right]e^{j\omega t}\,d\omega = \sin(\omega_o t)$

5.12 Find the Fourier transform of the following signals.
(a) $x_1(t) = e^{-3t}\sin 10tu(t)$
(b) $x_2(t) = e^{-4t}\cos 10tu(t)$

5.13 (a) Find the Fourier transform of $x(t) = 2\cos(\omega_o t + \theta)$
(b) Use the result in part (a) to derive the Fourier transform of $y(t) = 2\cos(100\pi t - \pi/4)$.

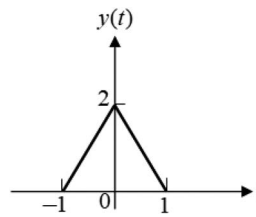


FIGURE 5.26 For Problem 5.8.

5.14 Given that $F(\omega) = \mathcal{F}[f(t)]$, prove the following results using the definition of Fourier transform:

(a) $\mathcal{F}[f(t-t_o)] = e^{-j\omega t_o} F(\omega)$

(b) $\mathcal{F}\left[\dfrac{df(t)}{dt}\right] = j\omega F(\omega)$

(c) $\mathcal{F}[f(-t)] = F(-\omega)$

(d) $\mathcal{F}[tf(t)] = j\dfrac{d}{d\omega}F(\omega)$

5.15 A signal has the Fourier transform

$$X(\omega) = \frac{5 + j\omega}{-\omega^2 + j2\omega + 1}$$

Determine the Fourier transform of these signals:

(a) $y(t) = x(2t - 5)$

(b) $y(t) = tx(t)$

(c) $y(t) = x(t)\sin 2t$

(d) $y(t) = \dfrac{d^2x(t)}{dt^2}$

5.16 A signal $f(t)$ has a Fourier transform given by

$$F(\omega) = \frac{5(1 + j\omega)}{8 - \omega^2 + 6j\omega}$$

Without finding $f(t)$, find the Fourier transform of the following:

(a) $f(t - 3)$

(b) $f(4t)$

(c) $e^{-j2t} f(t)$

(d) $f(-2t)$

5.17 Let the Fourier transform of $x(t)$ be

$$X(\omega) = \frac{j\omega}{4 + j\omega}$$

Find the Fourier transform of the following signals:

(a) $x(t - 2)$, (b) $x(3t)$, (c) $x'(t) = dx/dt$

5.18 The Fourier transform of a signal $x(t)$ is

$$X(\omega) = \frac{2}{j\omega} + \pi\delta(\omega)$$

Derive the Fourier transform of the following signals:

(a) $x_1(t) = 4x(2t - 1)$

(b) $x_2(t) = x(t - 2)\sin t$

(c) $x_3(t) = \dfrac{d}{dt}x(t)$

5.19 Obtain the Fourier transform of each of the following signals;

(a) $x(t) = 4\cos 2t + 2\delta(t)$

(b) $y(t) = 4\sin 2t - 2\sin 2(t-1) - 2\sin 2(t+1)$

5.20 If $X(\omega)$ is the Fourier transform of $x(t)$, show that

$$x(t+a) + x(t-a) \Leftrightarrow 2X(\omega)\cos a\omega$$

5.21 Use the convolution property to derive the Fourier transform of the time-domain product $y(t) = \sin 2t \cdot u(t)$

5.22 Let a signal $x(t)$ have the Fourier transform $X(\omega)$ shown in Figure 5.27. Without computing $x(t)$, sketch $Y(\omega)$ when:

(a) $y(t) = x(2t)$

(b) $y(t) = x(t/4)$

(c) $y(t) = x(t)\cos 2t$

(d) $y(t) = x(t)\sin 2t$

5.23 Find the Fourier transform of the following signals:

(a) $x(t) = \delta(t-2) + \delta(t-1) + \delta(t+1) + \delta(t+2)$

(b) $y(t) = 4te^{-t}u(t)$

(c) $z(t) = 2e^{-t}\sin(4t)u(t)$

5.24 Verify the Fourier transform pairs:

$$te^{-at}u(t) \Leftrightarrow \frac{1}{(a+j\omega)^2}, \quad a > 0$$

5.25 Verify the following identities:

(a) $X(0) = \int_{-\infty}^{\infty} x(t)dt$

(b) $x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$

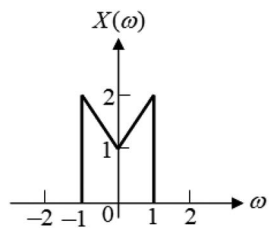


FIGURE 5.27 For Problem 5.22.

5.26 (a) A signal $x(t)$ has the equivalent bandwidth B and equivalent duration T defined as

$$T = \frac{\int_{-\infty}^{\infty} x(t)dt}{x(0)}, \quad B = \frac{\int_{-\infty}^{\infty} X(\omega)d\omega}{X(0)}$$

Show that $BT = 2\pi$.

(b) Determine the equivalent bandwidth of (i) $\Pi(t/\tau)$, (ii) $\Lambda(t/\tau)$.

5.27 If $X(\omega) = \mathcal{F}[x(t)]$ and a and b are constants, prove the following identities:

(a) $\mathcal{F}\left[x\left(\frac{t}{a}+b\right)\right] = |a|e^{jab\omega}X(\omega)$

(b) $\mathcal{F}[x(t)\delta(t-a)] = X(a)e^{-j\omega a}$

(c) $\mathcal{F}\left[x(t) * \delta\left(\frac{t}{a}-b\right)\right] = |a|e^{-jab\omega}X(\omega)$

5.28 Let $x(t)$ be given by $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) Sketch $x(t)$. (b) Obtain $X(\omega)$ by using differentiation property.

5.29 (a) If $x(t)$ is real and even, show that $X(\omega)$ is real and even.

(b) Give an example for part (a).

5.30 If $x(t)$ and $y(t)$ are two signals with Fourier transforms $X(\omega)$ and $Y(\omega)$, show that

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$$

5.31 Consider $X(\omega)$ as the Fourier transform of $x(t)$ in Figure 5.28. Evaluate:

(a) $X(0)$

(b) $\int_{-\infty}^{\infty} X(\omega)d\omega$

(c) $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

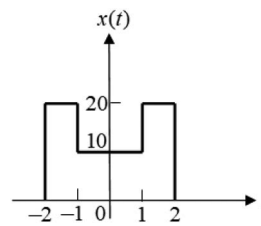


FIGURE 5.28 For Problem 5.31.

5.32 Evaluate the Fourier transform of

$$x(t) = \begin{cases} \sin t, & 0 < t < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

5.33 Find the Fourier transform of:
(a) $te^{-at}u(t)$, (b) $t^n e^{-at}u(t)$

Sections 5.4: Inverse Fourier Transform

5.34 Find $x(t)$ for each of the following:

(a) $X(\omega) = \dfrac{4\pi\delta(\omega - 1)}{2 + j\omega}$

(b) $X(\omega) = 4\pi\delta(\omega) - \dfrac{2}{j\omega}$

(c) $X(\omega) = 10\sin c(4\omega)$

5.35 Find the signal corresponding to each of the following Fourier transforms:

(a) $X(\omega) = \dfrac{2j\omega}{1 + j\omega}$

(b) $Y(\omega) = 10e^{-|\omega|}$

(c) $Z(\omega) = \dfrac{j\omega}{-\omega^2 + j5\omega + 6}$

5.36 Determine the inverse Fourier transform of

$$Y(\omega) = \frac{10}{3 - \omega^2 + j4\omega}$$

5.37 Determine the inverse Fourier transform of the signal in Figure 5.29.

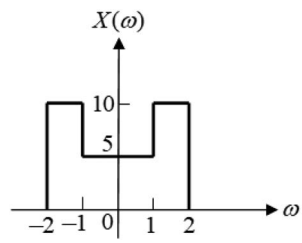


FIGURE 5.29 For Problem 5.37.

Section 5.4: Applications

5.38 The transfer function of a system is

$$H(\omega) = \frac{10}{2 + j\omega}$$

Determine the response $y(t)$ for the following inputs:
(a) $x(t) = 5\cos(2t)$
(b) $x(t) = \delta(t)$
(c) $x(t) = e^{-4t}\,u(t)$

5.39 A signal $x(t)$ is passed through a system as shown in Figure 5.30.
(a) Find the transfer function $H(\omega)$.
(b) If $x(t) = \cos(t)$, find $y(t)$.
(c) If $x(t) = e^{-3t}\,u(t)$, find $y(t)$.

5.40 The transfer function of a circuit is

$$H(\omega) = \frac{2}{j\omega + 2}$$

If the input signal to the circuit is $v_s(t) = e^{-4t}u(t)$, find the output signal. Assume all initial conditions are zero.

5.41 Apply Fourier transform analysis to find $v_o(t)$ in the circuit in Figure 5.31. Let $v_i(t) = 4e^{-2t}$ V.

5.42 If $i_s(t) = 4e^{-3t}u(t)$, find $i_o(t)$ in the circuit shown in Figure 5.32.

5.43 Calculate $i_o(t)$ in the circuit in Figure 5.33. Let $v_s(t) = 10e^{-3t}u(t)$ V.

5.44 Use the Fourier transform to find $i(t)$ in the circuit of Figure 5.34 if $v_s(t) = 10e^{-2t}u(t)$.

5.45 Find the voltage $v_o(t)$ in the circuit of Figure 5.35. Let $i_s(t) = 8e^{-t}u(t)$ A.

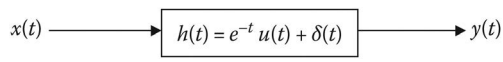


FIGURE 5.30 For Problem 5.39.

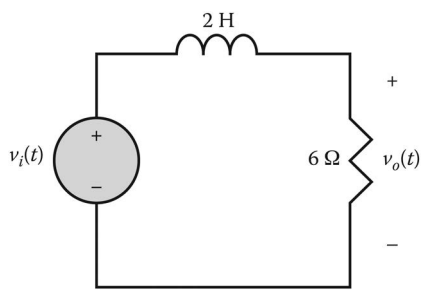


FIGURE 5.31 For Problem 5.41.

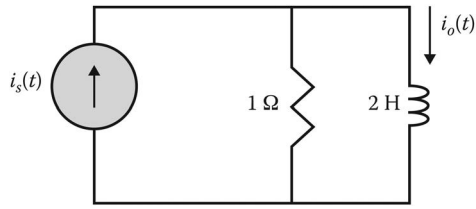


FIGURE 5.32 For Problem 5.42.

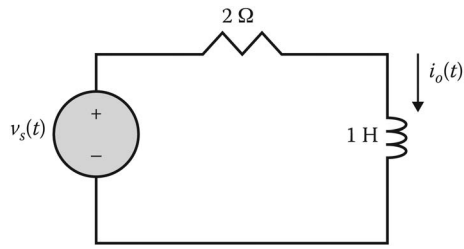


FIGURE 5.33 For Problem 5.43.

5.46 For the circuit in Figure 5.36, find the transfer function $H(\omega) = V_o(\omega)/V_i(\omega)$.

5.47 Obtain the response $v_o(t)$ of the circuit in Figure 5.37, where $v_s(t) = e^{-2t}u(t)$ V.

5.48 Refer to the filter shown in Figure 5.38. Find the transfer function and determine the type of filter.

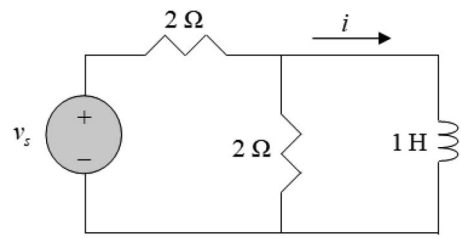


FIGURE 5.34 For Problem 5.44.

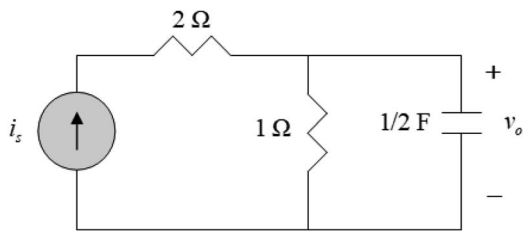


FIGURE 5.35 For Problem 5.45.

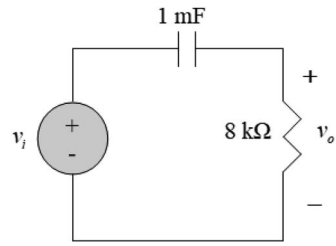


FIGURE 5.36 For Problem 5.46.

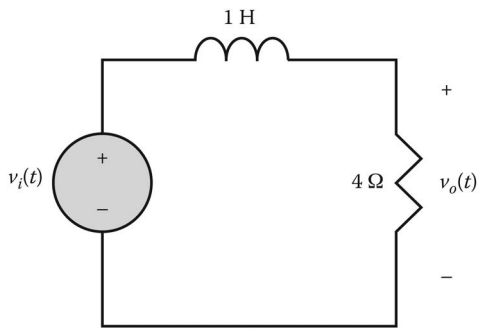


FIGURE 5.37 For Problem 5.47.

5.49 A continuous-time linear system is described by

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Find the output $y(t)$ due to the input $x(t) = e^{-2t}u(t)$.

5.50 Determine the transfer function of the linear system characterized by

$$\frac{d^2y(t)}{dt^3} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

and find its impulse response.

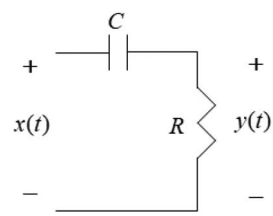


FIGURE 5.38 For Problem 5.48.

5.51 Find the response of a system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t), \quad x(t) = e^{-t}u(t)$$

using the Fourier transform.

5.52 A linear system is described by

$$\frac{dx(t)}{dt} + x(t) = e^{-2t}u(t)$$

Use Fourier transform to find $x(t)$.

5.53 In a system, the input signal $x(t)$ is amplitude modulated by $m(t) = 2 + \cos\omega_o t$. The response $y(t) = m(t)x(t)$. Find $Y(\omega)$ in terms of $X(\omega)$.

5.54 In a communication system, two signals

$$x_1(t) = \cos(100\pi t) \quad \text{and} \quad x_2(t) = 2\cos(400\pi t)$$

are multiplied to give $x_3(t) = x_1(t)x_2(t)$. Sketch the frequency spectra of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

5.55 Determine the Nyquist rates for these signals:

(a) $x(t) = 10\cos(60\pi t)$

(b) $x(t) = 3\cos(40\pi t) + 2\sin(50\pi t)$

5.56 A signal $x(t)$ is passed through a lowpass filter. If the filter blocks frequencies above 6 Hz, find the output $y(t)$ for the following cases:

(a) $x(t) = 2 - 3\cos(2\pi t) + 4\sin(13\pi t) + 10\cos(15\pi t)$

(b) $x(t) = 4 - 2\sin^2(4\pi t)$.

Section 5.5: Parseval's Theorem

5.57 Find the energy dissipated by the resistor in the circuit of Figure 5.39.

5.58 Calculate the total energy of the signal whose magnitude spectrum is shown in Figure 5.40.

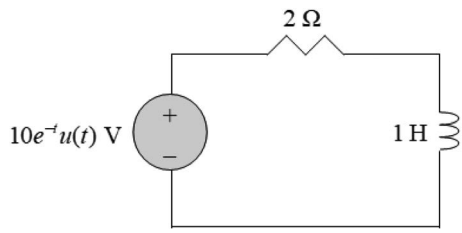


FIGURE 5.39 For Problem 5.57.

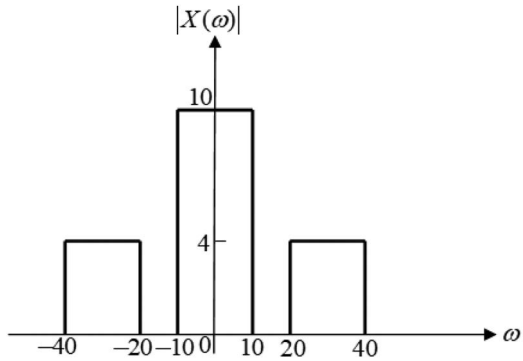


FIGURE 5.40 For Problem 5.58.

5.59 If $x(t)$ and $X(\omega)$ are Fourier pairs and

$$X(\omega) = \begin{cases} 10, & |\omega| < 2 \\ 0, & |\omega| > 2 \end{cases}$$

Let $y(t) = \dfrac{d^2x(t)}{dt^2}$. Find $\displaystyle\int_{-\infty}^{\infty} |y(t)|^2\, dt$.

Section 5.7: Computing with MATLAB

5.60 Use MATLAB to obtain the Fourier transform of the following signals.
(a) $x(t) = t\exp(-t)u(t)$
(b) $y(t) = (e^{-2t} - e^{-t})\,u(t)$
(c) $z(t) = \delta(t + 1/2) + e^{-t}\,u(t)$

5.61 Use MATLAB to computer the Fourier transform of

$$x(t) = 10e^{-2t}\cos(4t)u(t)$$

Plot the magnitude $|X(\omega)|$.

5.62 Use MATLAB to find the inverse Fourier transform of:

(a) $X_1(\omega) = \dfrac{e^{j\omega}}{1 - j\omega}$

(b) $X_2(\omega) = \dfrac{\delta(\omega)}{1 + j2\omega}$

(c) $X_3(\omega) = \dfrac{\pi\delta(\omega)}{(2 + j\omega)(5 + j\omega)}$

5.63 A system has the transfer function

$$H(s) = \frac{s^2 + s + 1}{2s^3 + 10s^2 + 16s + 8}, \quad s = j\omega$$

Use MATLAB to plot the frequency response.

5.64 Use MATLAB to plot the transfer function

$$H(\omega) = \frac{100(1 + j0.01\omega)}{(1 + j0.02\omega)(1 + j0.05\omega)}$$

5.65 The three-pole Butterworth filter has the transfer function

$$H(s) = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}, \quad s = j\omega$$

Use MATLAB to sketch $H(s)$ for $\omega_c = 2\pi$.

5.66 Use MATLAB to plot the transfer function

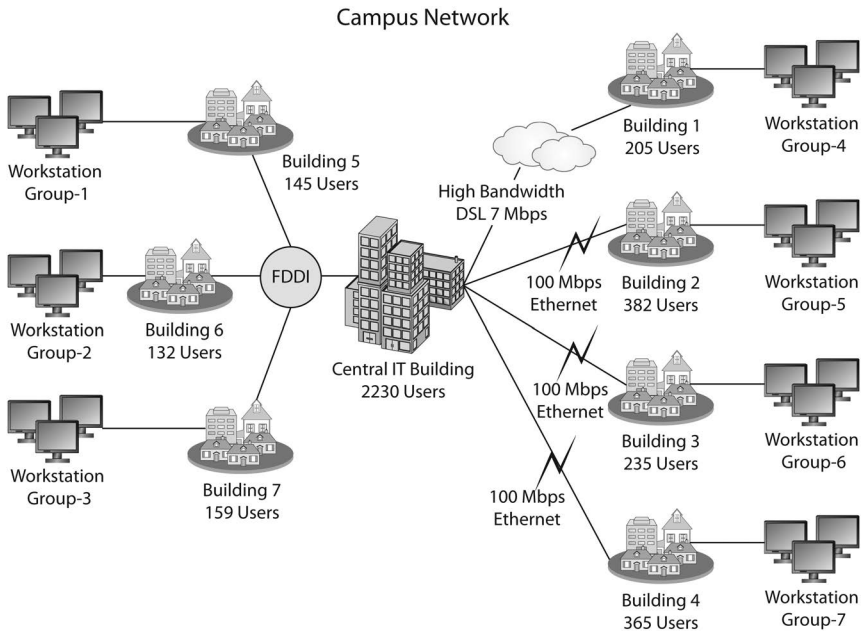
$$H(\omega) = \frac{1}{1 + j0.2\omega - 0.05\omega^2}$$

6 Discrete Fourier Transform

My philosophy of life is that if we make up our mind what we are going to make of our lives, then work hard toward that goal, we never lose—somehow we win out.

Ronald Reagan

CAMPUS COMMUNICATION NETWORK



Communications systems apply the principles of signals and systems. A communication system is designed to convey information from a source (the transmitter) to a destination (a receiver) via a channel (the propagation medium). Communication engineers design systems for transmitting and receiving information. The information can be in the form of voice, data, or video.

We live in the information age—news, weather, sports, shopping, financial, business inventory, and other sources make information available to us almost instantly via communications systems. Some obvious examples of communication systems are the telephone network, mobile cellular telephones, radios, cable TV, fax, and radar.

The field of communications is perhaps the fastest-growing area in electrical engineering. The merging of the communications field with computer technology

DOI: 10.1201/9781003490869-6

in recent years has led to digital data communication networks such as local area networks, metropolitan area networks, and broadband integrated services digital networks. For example, the Internet (the "information superhighway") allows educators, business people, and others to send electronic mail from their computers worldwide, log onto remote databases, and transfer files.

A communications engineer designs systems that provide high-quality information services. The systems include hardware for generating, transmitting, and receiving information signals. More and more government agencies, academic departments, and businesses are demanding faster and more accurate transmission of information. To meet these needs, communication engineers are in high demand.

6.1 INTRODUCTION

In the previous chapters, we have discussed applying frequency-domain techniques to continuous-time signals and systems. In this and the next chapter, we will consider frequency-domain analysis of discrete-time systems. Here, we specifically consider the Fourier analysis of discrete-time systems. Fourier analysis plays the same important role in discrete-time systems as in continuous-time systems.

Our approach is similar to that used for continuous-time signals and systems. We first introduce the discrete-time Fourier transform (DTFT) and its properties. We then discuss discrete Fourier transform (DFT), and the fast Fourier transform (FFT), a fast method of computing the DFT. The DFT plays a vital role in signal processing for spectral analysis, while FFT is the most commonly used Fourier analysis technique. The main advantage of DFT and FFT is that they are well suited for computation by a digital computer. We will learn how to use MATLAB® to implement numerically the discrete Fourier analysis covered in this chapter. We will finally apply the concepts learned in the chapter to the touch-tone telephone and windowing technique.

6.2 DISCRETE-TIME FOURIER TRANSFORM

In the last chapter, we defined the Fourier transform of a continuous-time signal $x(t)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{6.1}$$

By replacing the integral with summation, we obtain the *discrete-time Fourier transform* (DTFT) of $x[n]$ as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{6.2}$$

where we have used the uppercase omega (Ω) as the frequency variable to distinguish the continuous-time and discrete-time cases. $X(\Omega)$ is referred to as the Fourier spectrum of $x[n]$.

If we introduce $\Omega + 2\pi$ in eq. (6.2), we find that

$$X(\Omega + 2\pi) = X(\Omega), \quad -\infty < \Omega < \infty \tag{6.3}$$

since $e^{-j2\pi} = 1$. This shows that $X(\Omega)$ is periodic. For this reason, $X(\Omega)$ can be expanded in a Fourier series. If we multiply both sides of eq. (6.2) by $e^{j\Omega m} d\Omega$ and integrate, we obtain

$$\int_{-\pi}^{\pi} X(\Omega) e^{j\Omega m} d\Omega = \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right] e^{j\Omega m} d\Omega \tag{6.4}$$

But we know that the integral of any sinusoidal signal is zero when it is integrated over a period. Hence,

$$\sum_{n=-\infty}^{\infty} x[n] \int_{-\pi}^{\pi} e^{j\Omega(m-n)} \, d\Omega = \begin{cases} 2\pi x[n], & m = n \\ 0, & m \neq n \end{cases} \tag{6.5}$$

That is, only one term in the summation in eq. (6.4) remains. Thus, eq. (6.4) becomes

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} \, d\Omega \tag{6.6}$$

which is the inverse discrete-time Fourier transform (IDTFT)

From eqs. (6.2) and (6.6), we have the DTFT pair as:

$$\text{DFT: } X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \tag{6.7}$$

$$\text{IDFT: } x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \tag{6.8}$$

where $\mathcal{F}\{x[n]\}$ denotes the DTFT of $x[n]$. The one-to-one relationship between $x[n]$ and $X(\Omega)$ is symbolized as

$$x[n] \xleftrightarrow{\text{DTFT}} X(\Omega) \tag{6.9}$$

Since $X(\Omega)$ is periodic, the integral in eq. (6.8) can be evaluated over any interval of length 2π. For example,

$$x[n] = \frac{1}{2\pi}\int_{0}^{2\pi} X(\Omega)e^{j\Omega n}d\Omega \tag{6.10}$$

The sufficient condition for $X(\Omega)$ to exist (or to converge) is that $x[n]$ is absolutely summable; i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{6.11}$$

We should keep in mind that although ω has units of radians/second, Ω has units of radians. Although $x[n]$ is discrete, $X(\Omega)$ is a function of the continuous variable Ω.

Example 6.1

Given the discrete-time signal

$$x[n] = \begin{cases} 1, & n = -1 \\ -2, & n = 0 \\ 0, & n = 1 \\ 3, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the discrete-time Fourier transform.

SOLUTION

Since $x[n]$ has few nonzero terms, the finite sum becomes a sum of few terms.

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \\ &= e^{j\Omega} - 2\,e^{j0} + 0e^{j\Omega} + 3e^{-j2\Omega} \\ &= e^{j\Omega} - 2 + 3e^{-j2\Omega} \end{aligned}$$

Practice Problem 6.1 Find the DTFT of the signal

$$x[n] = \begin{cases} 2, & n = -2 \\ 3, & n = 0 \\ 1, & n = 1 \\ -1, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Answer: $2e^{j2\Omega} + 3 + e^{-j\Omega} - e^{-j2\Omega}$.

Example 6.2

Find the DTFT of the discrete-time signal

a. $x[n] = \delta[n]$

b. $x[n] = \begin{cases} a^n, & 0 \le n \le m \\ 0, & \text{otherwise} \end{cases}$

where a is a constant.

SOLUTION

To find the DTFT of $x[n]$, we apply eq. (6.7).

a. $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{m} \delta[n]e^{-j\Omega n} = e^{-j\Omega n}\Big|_{n=0} = 1$$

Thus,

$$\delta[n] \xleftrightarrow{\text{DTFT}} 1$$

b. $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{m} a^n e^{-j\Omega n} = \sum_{n=0}^{m} (ae^{-j\Omega})^n \tag{6.2.1}$$

But the geometric series sums as

$$\sum_{n=0}^{m} q^n = \frac{1-q^{m+1}}{1-q}, \quad |q| < 1 \tag{6.2.2}$$

with the common factor q. By letting $q = ae^{-j\Omega}$, we combine eqs. (6.2.1) and (6.2.2) to get

$$X(\Omega) = \frac{1-(ae^{-j\Omega})^{m+1}}{1-ae^{-j\Omega}}, \quad |a| < 1$$

Practice Problem 6.2 Determine the DTFT of the discrete-time signal $x[n] = a^n u[n]$, i.e.,

$$x[n] = \begin{cases} a^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

Answer: $\dfrac{1}{1-ae^{-j\Omega}}, |a| < 1.$

Example 6.3

Obtain the DTFT of the signal $x[n] = a^{|n|}$, $|a| < 1$.

SOLUTION

The Fourier transform is

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\Omega n} = \\ = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} + \sum_{n=-\infty}^{-1} a^{-n}e^{-j\Omega n} \tag{6.3.1}$$

If we make the substitution $m = -n$ in the second summation, we get

$$X(\Omega) = \sum_{n=0}^{\infty} (ae^{-j\Omega})^n + \sum_{m=1}^{\infty} (ae^{j\Omega})^m \tag{6.3.2}$$

Both summations are infinite geometric series

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots = \frac{1}{1-q}, \quad |q| < 1 \tag{6.3.3}$$

The first series in eq. (6.3.2) starts from $n = 0$, and its summation is given by eq. (6.3.3). However, the second series starts from $m = 1$. We need to subtract 1 from both sides of eq. (6.3.3) so that

$$\sum_{n=1}^{\infty} q^n = q + q^2 + \cdots = \frac{1}{1-q} - 1 = \frac{q}{1-q}, \quad |q| < 1 \tag{6.3.4}$$

Hence, applying eqs. (6.2.3) and (6.2.4) into eq. (6.2.2) gives

$$X(\Omega) = \frac{1}{1-ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1-ae^{j\Omega}} = \frac{1-a^2}{1-2a\cos\Omega + a^2} \tag{6.3.5}$$

Practice Problem 6.3 Find the DTFT of $a^n\, u[-(n+1)]$.

Answer: $\dfrac{1}{ae^{-j\Omega} - 1}, |a| < 1.$

Example 6.4

Obtain the DTFT of the constant signal $x[n]$ for all n. That is,

$$x[n] = 1, \quad n = 0, \pm 1, \pm 2, \ldots$$

SOLUTION

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} (1)e^{-jn\Omega} \tag{6.4.1}$$

By expanding the frequency domain train of impulse delta function into Fourier series, we can show that

$$\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) = \frac{1}{\omega_0} \sum_{n=-\infty}^{\infty} e^{-jnT_0\omega}, \quad T_0 = \frac{2\pi}{\omega_0}$$

or

$$\sum_{n=-\infty}^{\infty} e^{-jn\Omega} = 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - n2\pi) \tag{6.4.2}$$

where we have used $\Omega = T_0\omega$ and the property $\delta(at) = \frac{1}{|a|}\delta(t)$. Substituting eq. (6.4.2) in eq. (6.4.1) leads to the DTFT pair

$$x[n] = 1 \text{ for all } n \leftrightarrow 2\pi \sum_{n=-\infty}^{\infty} \delta(\Omega - 2\pi n)$$

Practice Problem 6.4 Obtain the DTFT of

$$x[n] = e^{j\Omega_0 n}$$

Answer: $2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi k)$.

6.3 PROPERTIES OF DTFT

The DTFT has some properties similar to those of the continuous-time Fourier transform. For each property, we shall state it and derive it and then illustrate with some examples.

6.3.1 Linearity

The linearity property of the DTFT states that if

$$x_1[n] \xleftrightarrow{\text{DTFT}} X_1(\Omega)$$

and

$$x_2[n] \xleftrightarrow{\text{DTFT}} \mathrm{X}_2(\Omega)$$

then

$$ax_1[n] + bx_2[n] \xleftrightarrow{\text{DTFT}} \mathrm{aX}_1(\Omega) + b\mathrm{X}_2(\Omega) \tag{6.12}$$

where a and b are constants. The proof is trivial.

Let us apply the linearity property to find the DTFT of the sinusoidal sequence

$$x[n] = \cos \Omega_0 n, \quad |\Omega_0| \leq \pi \tag{6.13}$$

Using Euler's formula,

$$\cos \Omega_0 n = \frac{1}{2}\left(e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right)$$

Applying the result in eq. (6.3.2) and the linearity property,

$$X(\Omega) = \pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right] \tag{6.14a}$$

that is,

$$\cos \Omega_0 n \xleftrightarrow{\text{DTFT}} \pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right] \tag{6.14b}$$

6.3.2 Time Shifting and Frequency Shifting

A delay or advance in the time domain results in a phase shift in the frequency domain. That is, if

$$x[n] \xleftrightarrow{\text{DTFT}} \mathrm{X}(\Omega)$$

then

$$x[n - n_0] \xleftrightarrow{\text{DTFT}} \mathrm{e}^{-\mathrm{j}\Omega \mathrm{n}_0}\mathrm{X}(\Omega) \tag{6.15}$$

This property can be proved using eq. (6.7).

$$\mathcal{F}\left\{x[n - n_0]\right\} = \sum_{n=-\infty}^{\infty} x[n - n_0]e^{-j\Omega n} \tag{6.16}$$

Let $n - n_0 = k$ on the right side of this equation so that $n = k + n_0$.

$$\begin{aligned}\mathcal{F}\{x[n-n_0]\} &= \sum_{n=-\infty}^{\infty} x[k]e^{-j\Omega(k+n_0)} = e^{-j\Omega n_0}\sum_{n=-\infty}^{\infty} x[k]e^{-j\Omega k} \\ &= e^{-j\Omega n_0}X(\Omega)\end{aligned} \tag{6.17}$$

For example, suppose we want to obtain the DTFT of the sequence $x[n] - x[n - 1]$. Using the time-shifting property,

$$x[n]-x[n-1] \xleftrightarrow{\text{DTFT}} (1-e^{-j\Omega})X(\Omega) \tag{6.18}$$

The dual property is the frequency shifting property, which states that

$$x[n]e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} \text{X}(\Omega-\Omega_0) \tag{6.19}$$

This can be derived by using eq. (6.7).

$$x[n]e^{j\Omega_0 n} \xleftrightarrow{\text{DTFT}} \sum_{\text{n}=-\infty}^{\infty}\text{x[n]}e^{j\Omega_0 n}e^{-j\Omega n} = \sum_{\text{n}=-\infty}^{\infty}\text{x[n]}e^{-j(\Omega-\Omega_0)} = X(\Omega-\Omega_0) \tag{6.20}$$

From this, it follows that

$$x[n]e^{-j\Omega_0 n} \xleftrightarrow{\text{DTFT}} \text{X}(\Omega+\Omega_0) \tag{6.21}$$

Combining eqs. (6.18) and (6.20), we get

$$x[n]\cos\Omega_0 n \xleftrightarrow{\text{DTFT}} \frac{1}{2}\left[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)\right] \tag{6.22}$$

which is the modulation property

6.3.3 Time Reversal and Conjugation

If $x[n] \xleftrightarrow{\text{DTFT}} \text{X}(\Omega)$, then

$$x[-n] \xleftrightarrow{\text{DTFT}} \text{X}(-\Omega) \tag{6.23}$$

We can prove this using eq. (6.7).

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\Omega n}$$

If we let $-n = k$, then

$$\mathcal{F}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[k]e^{j\Omega k} = \sum_{n=-\infty}^{\infty} x[k]e^{-j(-\Omega)k} = X(-\Omega) \tag{6.24}$$

A related property is the conjugation of $x[n]$. That is,

$$x^*[n] \xleftrightarrow{\text{DTFT}} X^*(-\Omega) \tag{6.25}$$

where * denotes the complex conjugate.

6.3.4 Time Scaling

Time scaling for a discrete-time signal is different from time scaling for a continuous-time signal. We recall if $X(\omega)$ is the Fourier transform of the continuous-time signal $x(t)$, then

$$\mathcal{F}[x(at)] = \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{6.26}$$

In the case of discrete time signals, $x[an]$ is not a sequence if a is not an integer. To derive a result that closely parallels eq. (6.26), let k be a positive integer. We define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k] = x[m], & \text{if } n = km,\ m = \text{integer} \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases} \tag{6.27}$$

Since $x_{(k)}[n]$ is zero unless n is a multiple of k, the DTFF of $x_{(k)}[n]$ is given by

$$X_{(k)}(\Omega) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\Omega n} = \sum_{m=-\infty}^{\infty} x_{(k)}[mk]e^{-j\Omega mk}$$

Replacing $x_{(k)}[n]$ by $x[m]$ gives

$$X_{(k)}(\Omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j(k\Omega)m} = X(k\Omega) \tag{6.28}$$

That is,

$$x_{(k)}[n] \xleftrightarrow{\text{DTFT}} X(k\Omega) \tag{6.29}$$

This shows that as the signal spreads out in time ($k > 1$), the corresponding Fourier transform is compressed. Since $X(\Omega)$ is periodic with period 2π, $X(k\Omega)$ is periodic with period $2\pi/k$.

6.3.5 Frequency Differentiation

This property states that if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$nx[n] \xleftrightarrow{\text{DTFT}} j\frac{dX(\Omega)}{d\Omega} \tag{6.30}$$

This can be proved by differentiating both sides of eq. (6.7) with respect to Ω. We obtain

$$\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\Omega n}$$

Multiplying both sides by j gives

$$j\frac{dX(\Omega)}{d\Omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\Omega n} \tag{6.31}$$

The right-hand side of this equation is the Fourier transform of $nx[n]$. Thus, eq. (6.30) follows.

For example, suppose we know that

$$\mathcal{F}\left\{a^n u[n]\right\} = \frac{e^{j\Omega}}{e^{j\Omega} - a}, \quad |a| < 1 \tag{6.32}$$

Using eq. (6.30), we obtain

$$\mathcal{F}\left\{na^n u[n]\right\} = j\frac{d}{d\Omega}\left(\frac{e^{j\Omega}}{e^{j\Omega} - a}\right) = \frac{ae^{j\Omega}}{(e^{j\Omega} - a)^2} \tag{6.33}$$

6.3.6 Time and Frequency Convolution

If $x_1[n] \xleftrightarrow{\text{DTFT}} \mathrm{X}_1(\Omega)$ and $x_2[n] \xleftrightarrow{\text{DTFT}} \mathrm{X}_2(\Omega)$, then

$$x_1[n] * x_2[n] \xleftrightarrow{\text{DTFT}} \mathrm{X}_1(\Omega) X_2(\Omega) \tag{6.34}$$

To prove this, we substitute the definition of discrete-time convolution in eq. (6.7).

$$\mathcal{F}\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right) e^{-j\Omega n} \tag{6.35}$$

If we change the order of the summations, we obtain

$$\mathcal{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] \left(\sum_{n=-\infty}^{\infty} x_2[n-k] e^{-j\Omega n} \right) \tag{6.36}$$

Applying the time-shifting property,

$$\sum_{n=-\infty}^{\infty} x_2[n-k] e^{-j\Omega n} = e^{-i\Omega k} X_2(\Omega) \tag{6.37}$$

Substituting this in eq. (6.36) yields

$$\mathcal{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k] e^{-j\Omega k} X_2(\Omega) = X_1(\Omega) X_2(\Omega) \tag{6.38}$$

This confirms eq. (6.34) for time convolution.

For example, for a discrete-time system with input $x[n]$, output $y[n]$, and impulse response $h[n]$, we obtain

$$Y(\Omega) = H(\Omega) X(\Omega) \tag{6.39}$$

For frequency convolution,

$$x_1[n] x_2[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi} \mathrm{X}_1(\Omega) \otimes X_2(\Omega) \tag{6.40}$$

This is also called the *multiplication property* since multiplication is involved in the time domain. This property states the multiplication of two signals in the time

domain corresponds to periodic convolution in the frequency domain, scaled by the inverse of the period. Periodic convolution is defined by

$$X_1(\Omega) \otimes X_2(\Omega) = \int_0^{2\pi} X_1(\theta) X_2(\Omega - \theta) d\theta \tag{6.41}$$

To prove eq. (6.40), we have by definition

$$\mathcal{F}\{x_1[n]x_2[n]\} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_2[n]\left[\frac{1}{2\pi}\int_0^{2\pi} X_1(\tau)e^{-jn\tau}d\tau\right]e^{-j\Omega n} \tag{6.42}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned}\mathcal{F}\{x_1[n]x_2[n]\} &= \frac{1}{2\pi}\int_0^{2\pi} X_1(\tau)\left[\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\Omega-\tau)n}\right]d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} X_1(\tau)X_2(\Omega-\tau)d\tau = \frac{1}{2\pi}X_1(\Omega)\otimes X_2(\Omega)\end{aligned} \tag{6.43}$$

For example, we can derive the modulation property as a special case of the frequency convolution.

$$\mathcal{F}\left[e^{j\Omega_0 n}x[n]\right] = \frac{1}{2\pi}\left[2\pi\delta(\Omega-\Omega_0)\right]\otimes X(\Omega) = X(\Omega-\Omega_0) \tag{6.44}$$

6.3.7 Accumulation

Accumulation (or summation) is the discrete-time counterpart of integration. The accumulation property can be stated as

$$\sum_{k=-\infty}^{\infty} x[k] \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\Omega}}X(\Omega) + \pi X(0)\sum_{n=-\infty}^{\infty}\delta(\Omega-2n\pi) \tag{6.45}$$

The accumulation operation is defined by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \tag{6.46}$$

This is equivalent to convolution according to eq. (2.21),

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] = x[n] * u[n] \tag{6.47}$$

Applying the frequency-convolution property,

$$\begin{aligned} Y(\Omega) = X(\Omega)U(\Omega) &= X(\Omega)\left[\frac{1}{1-e^{-j\Omega}} + \pi\sum_{n=-\infty}^{\infty}\delta(\Omega - 2n\pi)\right] \\ &= \pi X(0)\sum_{n=-\infty}^{\infty}\delta(\Omega - 2n\pi) + \frac{1}{1-e^{-j\Omega}}X(\Omega) \end{aligned} \tag{6.48}$$

6.3.8 Parseval's Relation

This can be derived from the convolution property. The energy of a discrete-time signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{6.49}$$

We may regard this as the value of DTFT of the signal

$$g[n] = |x[n]|^2 = x[n]x^*[n] \tag{6.50}$$

Hence,

$$G(\Omega) = \frac{1}{2\pi}X(\Omega)X^*(-\Omega) \tag{6.51}$$

Thus,

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |X(\Omega)|^2 \, d\Omega \tag{6.52}$$

This is Parseval's relation. $|X(\Omega)|^2$ is the energy spectrum of the signal.

For convenience, Table 6.1 summarizes the properties of DTFT, while Table 6.2 presents the transforms of some common transform pairs.

TABLE 6.1
Properties of DTFT

Property	Time Domain	Frequency Domain
1. Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
2. Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\Omega) + bX_2(\Omega)$
3. Time-shifting	$x[n-k]$	$e^{-j\Omega k}X(\Omega)$
4. Frequency-shifting (modulation)	$e^{j\Omega_o n}x[n]$	$X(\Omega - \Omega_0)$
5. Time reversal	$x[-n]$	$X(-\Omega)$
6. Conjugation	$x^*[n]$	$X^*(-\Omega)$
7. Time scaling	$x_{(k)}[n]$	$X(k\Omega)$
8. Differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
9. Time-convolution	$x_1[n]^*x_2[n]$	$X_1(\Omega)X_2(\Omega)$
9. Frequency-convolution	$x_1[n]x_2[n]$	$\dfrac{1}{2\pi}X_1(\Omega) \otimes X_2(\Omega)$
10. Accumulation	$\sum_{n=-\infty}^{\infty} x[k]$	$\dfrac{1}{1-e^{-j\Omega}}X(\Omega) + \pi X(0)\sum_{n=-\infty}^{\infty}\delta(\Omega - 2n\pi)$
11. Parseval's relation	$E_x = \sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2$	$E_x = \dfrac{1}{2\pi}\int_0^{2\pi}\lvert X(\Omega)\rvert^2 d\Omega$

Example 6.5

Determine the DTFT of $x[n] = a^{|n|}$.

SOLUTION

$x[n]$ can be expressed in terms of $a^n u[n]$ and $a^{-n}u[-n]$; i.e.,

$$a^{|n|} = a^n u[n] + a^{-n}u[-n] - \delta[n] \tag{6.5.1}$$

We know from Table 6.3 that

$$\mathcal{F}\left\{a^n u[n]\right\} = \frac{1}{1 - ae^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad |a| \le 1 \tag{6.5.2}$$

Using the time reversal property,

$$\mathcal{F}\left\{a^{-n}u[-n]\right\} = \frac{e^{-j\Omega}}{e^{-j\Omega} - a} \quad |a| \le 1 \tag{6.5.3}$$

TABLE 6.2
Common DTFT Pairs

Signal	Type	$x[n]$	$X(\Omega)$
1.	Impulse	$\delta[n]$	1
2.	Shifted impulse	$\delta[n-k]$	$e^{-j\Omega k}$
3.	Unit step	$u[n]$	$\frac{1}{1-e^{-j\Omega}}+\pi\sum_{n=-\infty}^{\infty}\delta(\Omega-2n\pi), \quad \lvert\Omega\rvert\le\pi$
4.	Shifted unit step	$-u[-n-1]$	$\frac{1}{1-e^{-j\Omega}}-\pi\sum_{n=-\infty}^{\infty}\delta(\Omega-2n\pi), \quad \lvert\Omega\rvert\le\pi$
5.	DC signal	1, for all n	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$
6.	Gated function	$u[n]-u[n-k]$	$\frac{\sin(k\Omega/2)}{\sin(\Omega/2)}e^{-j\Omega(k-1)/2}$
7.	Exponential	$a^n u[n]$	$\frac{1}{1-ae^{-j\Omega}} \quad \lvert a\rvert\le 1$
8.	Weighted exponential	$na^n u[n]$	$\frac{ae^{-j\Omega}}{(1-ae^{-j\Omega})^2} \quad \lvert a\rvert\le 1$
		$(n+1)a^n u[n]$	$\frac{1}{(1-ae^{-j\Omega})^2} \quad \lvert a\rvert\le 1$
9.	Two-sided exponential	$a^{\lvert n\rvert}$	$\frac{1-a^2}{1+a^2-2a\cos\Omega} \quad \lvert a\rvert<1$
10.	Complex sinusoid	$e^{j\Omega_0 n}$	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_o-2\pi k)$
11.	Cosine wave	$\cos\Omega_0 n$	$\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_o-2\pi k)+\delta(\Omega+\Omega_o-2\pi k)$
12.	Sine wave	$\sin\Omega_0 n$	$\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(\Omega-\Omega_o-2\pi k)-\delta(\Omega+\Omega_o-2\pi k)$

Taking the DTFT of each term in eq. (6.5.1) and substituting eqs. (6.5.2) and (6.5.3) gives

$$\mathcal{F}\left\{a^{|n|}\right\}=\frac{e^{j\Omega}}{e^{j\Omega}-a}+\frac{e^{-j\Omega}}{e^{-j\Omega}-a}-1=\frac{1-a^2}{1-2a\cos\Omega+a^2} \quad |a|\le 1 \tag{6.5.4}$$

Practice Problem 6.5 Determine the DTFT of the composite signal

$$x[n]=a^n u[n]-\sin\Omega_0 n, \quad |a|<1$$

Answer: $\frac{1}{1-ae^{-j\Omega}}+j\pi\left[\delta(\Omega-\Omega_o)-\delta(\Omega+\Omega_o)\right]$.

Example 6.6

Given the signal

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & \text{otherwise} \end{cases} \tag{6.6.1}$$

a. sketch $x[n]$ and its DTFT $X(\Omega)$, (b) sketch the time-scaled signal $x_{(2)}[n]$ and its DTFT $X_{(2)}(\Omega)$.

SOLUTION

a. The rectangular pulse is shown in Figure 6.1(a). It is also known as a five-point rectangular window function. Its Fourier transform is

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-(N-1)/2}^{(N-1)/2} (1)(e^{-j\Omega})^n, \quad N = 5 \tag{6.6.2}$$

This is a geometric sequence with factor $e^{-j\Omega}$. From Appendix A.10,

$$\sum_{k=M}^{N} a^k = \frac{a^{N+1} - a^M}{a-1} \, a \neq 1 \tag{6.6.3}$$

Applying this to eq. (6.6.1) yields

$$\begin{aligned} X(\Omega) &= \frac{e^{-j[(N+1)/2]\Omega} - e^{j[(N-1)/2]\Omega}}{e^{-j\Omega} - 1} = \frac{e^{-j\Omega/2}\left(e^{-j(N/2)\Omega} - e^{j(N/2)\Omega}\right)}{e^{-j\Omega/2}(e^{-j\Omega/2} - e^{j\Omega/2})} \\ &= \frac{\sin(N\Omega/2)}{\sin(0.5\Omega)}, \quad N = 5 \\ &= \frac{\sin(2.5\Omega)}{\sin(0.5\Omega)} \end{aligned} \tag{6.6.4}$$

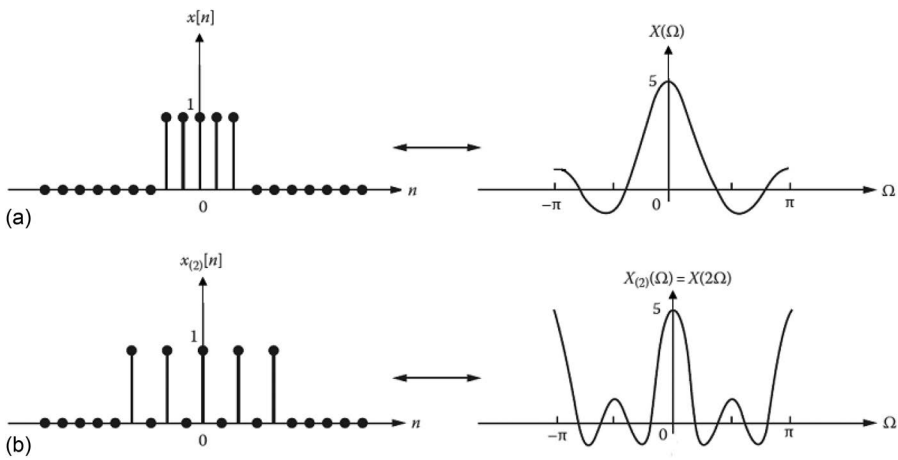


FIGURE 6.1 For Example 6.6.

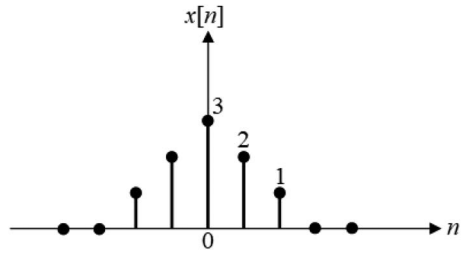


FIGURE 6.2 For Practice Problem 6.5.

The sequence in eq. (6.6.1) and its Fourier transform in eq. (6.6.4) are sketched in Figure 6.1(a). Notice that although $x[n]$ is discrete, $X(\Omega)$ is continuous.

b. Using the time-scaling property in eq. (6.29) and eq. (6.6.4), we obtain

$$X_{(2)}(\Omega) = X(2\Omega) = \frac{\sin(5\Omega)}{\sin(\Omega)}$$

Both the signal $x_{(2)}[n]$ and its DTFT $X_{(2)}(\Omega)$ are sketched in Figure 6.1(b). Notice that $x_{(2)}[n]$ is obtained from $x[n]$ by placing $k - 1 = 2 - 1$ zeros between successive values of the original signal $x[n]$.

Practice Problem 6.6 Find the DTFT of the sequence shown in Figure 6.2.

Answer: $3 + 2\cos 2\Omega + 4\cos\Omega$.

Example 6.7

Determine the inverse of the function

$$X(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^2}, \quad |a| < 1$$

SOLUTION

From Table 6.2,

$$a^n u[n] \xleftrightarrow{\text{DTFT}} \frac{1}{1 - ae^{-j\Omega}}, \quad |a| < 1$$

The given function can be expressed as

$$X(\Omega) = \frac{1}{(1 - ae^{-j\Omega})^2} = \left(\frac{1}{1 - ae^{-j\Omega}}\right)\left(\frac{1}{1 - ae^{-j\Omega}}\right)$$

By the convolution theorem,

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$= a^n \sum_{k=0}^{n} 1 = (n+1)a^n u[n], \quad |a| < 1$$

Practice Problem 6.7 Given that $h[n] = \left(\frac{1}{2}\right)^n u[n]$ and $x[n] = \left(\frac{1}{3}\right)^n u[n]$, find $y[n] = h[n]*u[n]$.

Answer: $3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$.

6.4 DISCRETE FOURIER TRANSFORM

The DFT may be regarded as a logical extension of the DTFT covered earlier. By sampling the DTFT $X(\Omega)$ at uniformly spaced frequencies $\Omega = 2\pi k/N$, where $k = 0,1,2,\ldots, N-1$, we can define the DFT of $x[n]$. Let $x[n]$, $n = 0,1,2,\ldots, N-1$, be an N-point sequence. We define the DFT of $x[n]$ as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \quad k = 0,1,2,...,N-1 \tag{6.53}$$

The DFT relates discrete-time sequence $x[n]$, $n = 0,1,2,\ldots, N-1$, to the discrete-frequency sequence $X[k]$, $k = 0,1,2,\ldots, N-1$. The DFT is defined by the pair

$$\text{DFT: } X[k] = \mathcal{F}[x[n]] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} \tag{6.54}$$

$$\text{IDFT: } x[n] = \mathcal{F}^{-1}[X[k]] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N} \tag{6.55}$$

Eq. (6.55) specifies the inverse DFT or IDFT. We use the notation

$$x[n] \xleftrightarrow{\text{DFT}} X[k] \tag{6.56}$$

TABLE 6.3
Properties of the DFT

Property	Time Domain	Frequency Domain
1. Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
2. Time-shifting	$x[n-m]$	$e^{-j2\pi km}X(k)$
3. Frequency-shifting (modulation)	$e^{-j2\pi k_0 n/N}x[n]$	$X(k-k_0)$
4. Time reversal	$x[-n]$	$X(-k)$
5. Conjugation	$x^*[n]$	$X^*(-k)$
6. Time-convolution	$x_1[n] \otimes x_2[n]$	$X_1[k]X_2[k]$
7. Frequency-convolution	$x_1[n]x_2[n]$	$\frac{1}{N}X_1[k] \otimes X_2[k]$
8. Parseval's relation	$E_x = \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$E_x = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k] \rvert^2$

Note that the DFT $X[k]$ as given in eq. (6.54) is a periodic sequence with period N. This should be expected since $X[k]$ is a sampled version of the periodic DTFT $X(\Omega)$. Also, note that both DFT and the IDFT begin the summation from zero and end at $N - 1$ so that both are finite. Since DFT can be derived from DTFT, we would expect their properties to be similar. The properties of DFT are listed in Table 6.3.

Notice that items 6 and 7 in Table 6.3 involve circular convolution. The convolution of two discrete-time signals covered in Chapter 2:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{6.57}$$

may be regarded as *linear convolution*. Consider two sequences, $x[n]$ and $h[n]$, defined for $n = 0, 1,\ldots, N - 1$; i.e., they both have length N and zero values of $n < 0$ and $n \geq N$. The *periodic or circular convolution* is defined as

$$y[n] = x[n] \otimes h[n] = \sum_{k=0}^{N-1} x[k]h[n-k] \tag{6.58}$$

It is often regarded as the evaluation of two signals around two concentric circles. Eq. (6.58) is evaluated by writing the N values of $x[n]$ equally spaced around an outer circle in a counterclockwise direction, while the N values of $h[n]$ are equally spaced in a clockwise direction on an inner circle.

In general, $X[k]$ is complex, and it may be easier to express it in terms of its real part $R[k]$ and imaginary part $I[k]$; i.e.,

$$X[k] = R[k] + jX[k] \tag{6.59}$$

where

$$R[k] = x[0] + \sum_{n=1}^{N-1} x[n] \cos\left(\frac{2\pi kn}{N}\right) \tag{6.60}$$

$$I[k] = -\sum_{n=1}^{N-1} x[n] \sin\left(\frac{2\pi kn}{N}\right) \tag{6.61}$$

Example 6.8

Find the DFT of the sequence $x[n] = \{1\ -2\ 1\ 3\}$.

SOLUTION

We can find the DFT of $x[n]$ either directly using eq. (6.54) or indirectly using eqs. (6.58) and (6.59).
$x[0] = 1$, $x[1] = -2$, $x[2] = 1$, $x[3] = 3$. With $N = 4$, eq. (6.54) becomes

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N} = \sum_{n=0}^{3} x[n] e^{-j\pi nk/2}, \quad k = 0,1,2,3 \\ &= x[0] + x[1]e^{-j\pi k/2} + x[2]e^{-j\pi k} + x[3]e^{-j\pi 3k/2}, \quad k = 0,1,2,3 \\ &= 1 - 2e^{-j\pi k/2} + e^{-j\pi k} + 3e^{-j\pi 3k/2}, \quad k = 0,1,2,3 \end{aligned} \tag{6.8.1}$$

The real part of this is

$$\begin{aligned} R[k] &= 1 - 2\cos(-\pi k/2) + \cos(-\pi k) + 3\cos(-3\pi k/2), \quad k = 0,1,2,3 \\ &= 1 - 2\cos(\pi k/2) + \cos(\pi k) + 3\cos(3\pi k/2), \quad k = 0,1,2,3 \end{aligned}$$

Evaluating this for each k leads to

$$R[k] = \begin{cases} 3, & k = 0 \\ 0, & k = 1 \\ 1, & k = 2 \\ 0, & k = 3 \end{cases} \tag{6.8.2}$$

The imaginary part of $X[k]$ is

$$\begin{aligned} I[k] &= -2\sin(-\pi k/2) + \sin(-\pi k) + 3\sin(-\pi 3k/2) \\ &= 2\sin(\pi k/2) - \sin(\pi k) - 3\sin(3\pi k/2), \quad k = 0,1,2,3 \end{aligned}$$

Evaluating this for each value of k, we get

$$I[k] = \begin{cases} 0, & k = 0 \\ 5, & k = 1 \\ 0, & k = 2 \\ -5, & k = 3 \end{cases} \tag{6.8.3}$$

From eqs. (6.8.2) and (6.8.3), we obtain $X[k] = R[k] + jI[k]$, i.e.,

$$X[k] = \begin{cases} 3, & k = 0 \\ j5, & k = 1 \\ 1, & k = 2 \\ -j5, & k = 3 \end{cases} \tag{6.8.4}$$

Practice Problem 6.8 Determine the DFT of sequence $x[n]$ = [5 –2 0 1 4] and its energy.

Answer: $X(k) = \begin{cases} 8, & k = 0 \\ 4.809 + j6.276, & k = 1 \\ 3.691 + j2.576, & k = 2, \\ 3.691 - j2.376, & k = 3 \\ 4.809 - j6.2971, & k = 4 \end{cases} \quad E_x = 46$

Example 6.9

Obtain the IDFT of $X(k)$ given by

$$X[0] = 5, \quad X[1] = -1 - 2j, \quad X[2] = -7, \quad X[3] = -1 + 2j$$

SOLUTION

We can solve this problem in two ways.

Method 1:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}$$

$$\text{For } n = 0, \quad x[0] = \frac{1}{4}(X[0] + X[1] + X[2] + X[3]) = \frac{1}{4}(5 - 1 - 2j - 7 - 1 + 2j) = -1$$

$$\text{For } n = 1, \quad x[1] = \frac{1}{4}(X[0] + X[1]e^{j\pi/2} + X[2]e^{j\pi} + X[3]e^{j3\pi/2}) = 4 \tag{6.9.1}$$

$$\text{For } n = 2, \quad x[2] = \frac{1}{4}(X[0] + X[1]e^{j\pi} + X[2]e^{j2\pi} + X[3]e^{j3\pi}) = 0$$

$$\text{For } n = 3, \quad x[3] = \frac{1}{4}(X[0] + X[1]e^{j3\pi/2} + X[2]e^{j3\pi} + X[3]e^{j9\pi/2}) = 2$$

The Euler's Formula $e^{j\theta} = \cos\theta + j\sin\theta$ *has been applied*.

Thus, $x[n]$ = [–1 4 0 2]

Method 2: Equation (6.9.1) could be regarded as the multiplication of two matrices.

$$e^{j2\pi kn/4} = \begin{matrix} \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \left\| \begin{matrix} e^0 & e^0 & e^0 & e^0 \\ e^0 & e^{j\pi/2} & e^{j\pi} & e^{j3\pi/2} \\ e^0 & e^{j\pi} & e^{j2\pi} & e^{j3\pi} \\ e^0 & e^{j3\pi/2} & e^{j3\pi} & e^{j9\pi/2} \end{matrix} \right\| \end{matrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -j \\ 1 & -j & -1 & j \end{bmatrix}$$

Thus,

$$x[n] = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -j \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 5 \\ -1-j2 \\ -7 \\ -1+j2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

Practice Problem 6.9 Determine the IDFT of the $X[k]$ obtained in Example 6.7.

Hint: Use eq. (6.55).

Answer: $x[n] = [1\ -2\ 1\ 3]$.

Example 6.10

Find the periodic convolution of the following two sequences.
$x[n] = [1, 0, -2, 3]$ and $h[n] = [3, 1, 2, -1]$

SOLUTION

$N = 4$ and we have a case similar to the previous example. Instead of using real and imaginary parts, we apply eq. (6.54) directly. From eq. (6.8.1),

$$X[k] = x[0] + x[1]e^{-j\pi k/2} + x[2]e^{-j\pi k} + [3]e^{-j\pi 3k/2}, \quad k = 0,1,2,3$$
$$X[0] = x[0] + x[1] + x[2] + x[3] = 2$$
$$X[1] = x[0] + x[1]e^{-j\pi/2} + x[2]e^{-j\pi} + x[3]e^{-j3\pi/2} = 3 + j3$$
$$X[2] = x[0] + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} = -4$$
$$X[3] = x[0] + x[1]e^{-j3\pi/2} + x[2]e^{-3j\pi} + x[3]e^{-j9\pi/2} = 3 - j3$$

Similarly,

$$H[0] = h[0] + h[1] + h[2] + h[3] = 5$$
$$H[1] = h[0] + h[1]e^{-j\pi/2} + h[2]e^{-j\pi} + h[3]e^{-j3\pi/2} = 1 - j2$$
$$H[2] = h[0] + h[1]e^{-j\pi} + h[2]e^{-j2\pi} + h[3]e^{-j3\pi} = 5$$
$$H[3] = h[0] + h[1]e^{-j3\pi/2} + h[2]e^{-3j\pi} + h[3]e^{-j9\pi/2} = 1 + j2$$

Convolution in the time domain produces multiplication in the frequency domain; i.e.,

$$y[n] = h[n] \otimes x[n] \leftrightarrow Y[k] = H(k)X(k)$$

$$Y[0] = H[0]X[0] = 10$$

$$Y[1] = H[1]X[1] = 9 - \text{j}3$$

$$Y[2] = H[2]X[2] = -20$$

$$Y[3] = H[3][3] = 9 + \text{j}3$$

We now use IDFT to find $y[n]$ from $Y[k]$.
From eq. (6.55),

$$y[n] = \mathcal{F}^{-1}[Y[k]] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi nk/N}$$

Since $N = 4$, we have

$$y[0] = \frac{1}{4}[Y[0] + Y[1] + Y[2] + Y[3]] = 2$$

$$y[1] = \frac{1}{4}\left[Y[0] + Y[1]e^{j\pi/2} + Y[2]e^{j\pi} + Y[3]e^{j3\pi/2}\right] = 9$$

$$y[2] = \frac{1}{4}\left[Y[0] + Y[1]e^{j\pi} + Y[2]e^{j2\pi} + Y[3]e^{j3\pi}\right] = -7$$

$$y[3] = \frac{1}{4}\left[Y[0] + Y[1]e^{j3\pi/2} + Y[2]e^{3j\pi} + Y[3]e^{j9\pi/2}\right] = 6$$

From this, we have $y[n] = [2, 9, -7, 6]$.

Practice Problem 6.10 Repeat Example 6.9 for

$$x[n] = [0, 1, 2, -3] \quad \text{and} \quad h[n] = \left(\frac{1}{2}\right)^n, \quad n = 0,1,2,3$$

Answer: $y[n] = [-0.875, 0.5, 2.125, -1.75]$.

Example 6.11

Verify Parseval's relation using the sequence $x[n] = \{1\ -2\ 1\ 3\}$.

SOLUTION

Using the given sequence directly, we get

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = 1 + 4 + 1 + 9 = 15$$

In Example 6.8, we obtained the DFT of the sequence as

$$X[k] = \begin{cases} 3, & k = 0 \\ j5, & k = 1 \\ 1, & k = 2 \\ -j5, & k = 3 \end{cases}$$

$$E_x = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{4}(9 + 25 + 1 + 25) = 15$$

confirming Parseval's relation.

Practice Problem 6.11 Repeat Example 6.9 for the sequence $x[n] = [0, 1, 2, -3]$.

Answer: $E_x = 14$.

6.5 FAST FOURIER TRANSFORM

The number of complex computations required in calculating DFT is drastically reduced by an algorithm known as the *fast Fourier transform* (FFT). The development of FFT, which dramatically reduces the number of computational operations, made DFT a very useful transform in many areas of applied science and technology. FFT was developed by James Cooley and John Tukey in 1965. Keep in mind that FFT is not a transform by itself; it is an algorithm—a fast computation of DFT.

For a discrete-time signal, we defined the N-point DFT and IDFT as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \quad k = 0, 1, ..., N-1 \tag{6.62}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N}, \quad n = 0, 1, 2, ..., N-1 \tag{6.63}$$

It is evident from eq. (6.62) that to compute each $X[k]$ requires N complex multiplications and $N - 1$ complex additions. Since we have to evaluate $X[k]$ for $k = 0, 1, 2, \ldots, N - 1$, the entire computation of DFT requires N^2 complex multiplications and $N(N - 1)$ complex additions. This becomes prohibitively time consuming for large N.

The numbers of complex multiplications required for N-point DFT (N^2) and FFT $\left(\frac{N}{2}\log_2 N\right)$ are compared in Table 6.4. We can see from the table that as N increases, the speed advantage of FFT becomes obvious. (For FFT computations,

TABLE 6.4
Number of Multiplications Required in DFT and FFT

m	$N = 2^m$	DFT	FFT
1	2	4	1
2	4	16	4
3	8	64	12
4	16	256	32
5	32	1,024	80
6	64	4,096	192
7	128	16,384	448
8	256	65,536	1,024
9	512	261,144	2,304
10	1,024	1,048,576	5,120

N need not be a power of 2.) The FFT algorithm is very useful in communications and digital signal processing.

Like IDFT, we also have inverse FTT (or IFFT). Although there are several variations of the Tukey-Cooley algorithm on FFT, they can be grouped into two types: decimation in time and decimation in frequency. Discussion of these algorithms is beyond the scope of this book. All that is needed at this stage is using FFT and IFFT in MATLAB.

6.6 COMPUTING WITH MATLAB®

MATLAB has in-built commands **fft** and **ifft** for finding, respectively, the FFT and IFFT of a discrete-time sequence. As we know, finding DFT is the same as finding FFT.

Example 6.11

A square wave is represented by the sequence $x[n]$ defined as

$$x[0] = x[1] = x[2] = x[3] = 1 \quad \text{and} \quad x[4] = x[5] = x[6] = x[7] = -1$$

Use MATLAB to find the FFT.

SOLUTION

We use the following MATLAB script to find $X[k]$.

```
x = [ 1 1 1 1 -1 -1 -1 -1];
X = fft(x)
stem(abs(X))
```

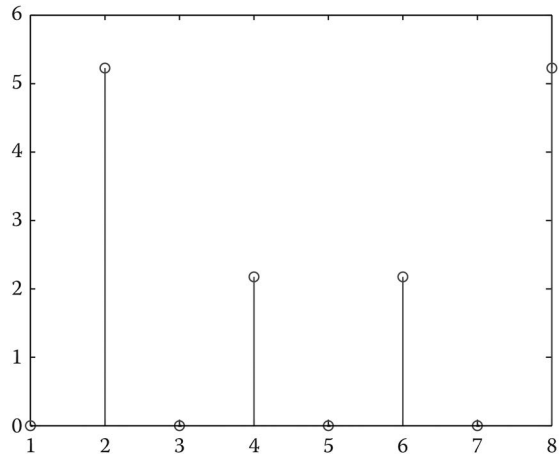


FIGURE 6.3 For Example 6.10.

This results in

```
X  = [ 0  2 - j4.8284  0  2 - j0.8284  0  2 + j0.8284 0 2 +
j4.8284]
```

The MATLAB command stem is used to plot the absolute value of the result, as shown in Figure 6.3. To ensure that the result is correct, we can find the IFFT of X.

```
X  = [0 2-j*4.8284 0 2-j*0.8284 0 2+j*0.8284 0 2+j*4.8284]
x = ifft(X)
```

The result is the same as the given $x[n]$.

Practice Problem 6.11 Find the FTT of the sequence $x[n] = \{1\ -2\ 1\ 3\}$.

Answer: [3, j5, 1, –j5].

Example 6.12

Repeat Example 6.9 using MATLAB.

SOLUTION

We find the FFT of sequences $x[n]$ and $h[n]$. We multiply the results together and then find IFFT to get $y[n]$. The MATLAB for doing the convolution follows.

```
x = [ 1  0 -2 3];
h = [ 3  1  2 -1];
X = fft(x);
H = fft(h);
Y = X.*H;
y = ifft(Y)
```

This produces $y[n] = [2, 9, -7, 6]$, which is what we got the hard way in Example 6.9.

Practice Problem 6.12 Repeat Practice Problem 6.9 using MATLAB.

Answer: $y[n] = [-0.875, 0.5, 2.125, -1.75]$.

Example 6.13

Although MATLAB does not have a function for DTFT, we can use it to plot the Fourier spectrum $X(\Omega)$. In Practice Problem 6.2, the DTFT of the discrete-time signal $x[n] = a^n u[n]$ is

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Plot this for $a = 0.8$.

SOLUTION

The necessary MATLAB script is shown next. The resulting plot of $X(\Omega)$ is in Figure 6.4.

```
a=0.8;
Omega= 0:0.01:2.0;
X =1./(1- a*exp(-j*Omega));
plot(Omega,abs(X));
xlabel('\Omega')
ylabel('X(\Omega)')
```

Practice Problem 6.13 Use MATLAB to plot the Fourier spectrum in eq. (6.3.5), namely,

$$X(\Omega) = \frac{ae^{j\Omega}}{(e^{j\Omega} - a)^2}, \quad a = 0.9$$

Answer: See Figure 6.5.

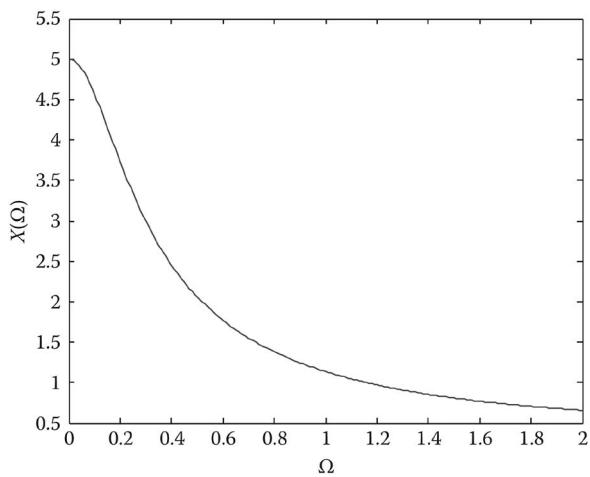


FIGURE 6.4 For Example 6.13.

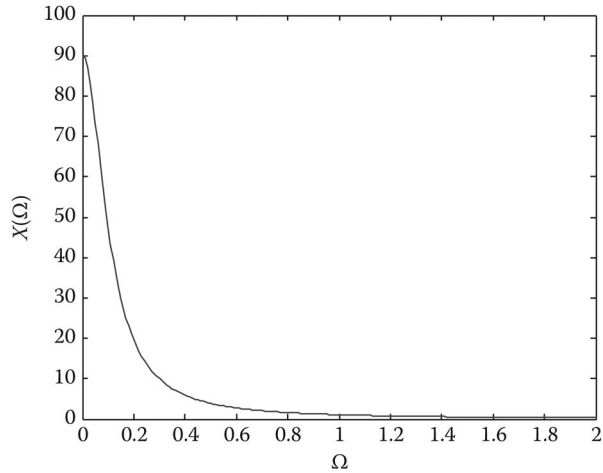


FIGURE 6.5 For Practice Problem 6.13.

6.7 APPLICATIONS

We will consider some applications of DFT. Specifically, we will consider the touch-tone telephone and windowing technique.

6.7.1 Touch-Tone Telephone

The touch-tone telephone uses signals of different frequencies to specify which button is being pushed. The keypad has 12 buttons arranged in four rows and three columns. The DTFT of a sampled signal can be used to identify these frequencies. Pushing a button generates a sum of two sinusoids corresponding to its unique pair of frequencies. The higher-frequency sinusoid indicates the column of the key, while the lower-frequency sinusoid indicates the row of the key. Table 6.5 shows the DTFT frequencies for touch-tone signals sampled at 8192 Hz. For example, the digit 8 is represented by the signal

$$x_8[n] = \sin(0.6535n) + \sin(1.0247n) \tag{6.64}$$

TABLE 6.5
DTFT Frequencies for Touch-Tone Signals

	ω_{column}		
ω_{row}	**0.9273**	**1.0247**	**1.1328**
0.5346	1	2	3
0.5906	4	5	6
0.6535	7	8	9
0.7217		0	

6.7.2 Windowing

Sampling a signal for storage in a computer requires that we limit the signal to a finite length. This truncation causes discontinuities at either end of the sampled signal. In order to minimize this effect, called *spectral leakage*, the data samples can be multiplied by windows which approach zero smoothly at the beginning and end of the signal.

Windowing is a fundamental operation in digital signal processing and is inherent in DFT development. It refers to the operation of multiplying a signal $x[n]$ by a finite-duration sequence $w[n]$ to produce a finite-duration signal $x_w[n]$; i.e.,

$$x_w[n] = x[n]w[n] \tag{6.65}$$

where

$$w[n] = 0 \text{ for } n < 0 \text{ and } n > N$$

This makes both the window sequence $w[n]$ and the windowed sequence $x_w[n]$ be of length N. A few examples of window sequences are:

1. Rectangular:

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.66}$$

2. Hanning:

$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right], \quad 0 \le n < \frac{N-1}{2} \tag{6.67}$$

3. Hamming:

$$w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \le n < N-1 \tag{6.68}$$

4. Barlett:

$$w[n] = 1 - \frac{2n}{N-1}, \quad 0 \le n < \frac{N-1}{2} \tag{6.69}$$

5. Blackman:

$$w[n] = 0.4 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \le n < N \tag{6.70}$$

The windowing technique is used in many applications. For example, it is used in Fourier analysis to decrease the size of the Gibbs overshoot and ripples. Windowing is also used in filter design. By using the windowing technique, we can obtain a filter that satisfies the design specs with fewer elements and less processing delay.

Example 6.14

Compute the DTFT of the discrete sequence $x[n] = 0.8^n u[n]$. Use $N = 32$ and compare the exact spectrum with the one obtained using the Hamming window.

SOLUTION

From Table 6.2, the exact DTFT of $u[n]$ is

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}} \quad |a| \le 1$$

where $a = 0.8$. We use the following MATLAB code for computing $X(\Omega)$ and the Hamming window.

```
a=0.8; N = 32;
n = 0:N-1
w = 0:0.01:pi; % values of Omega
fe = abs(1./(1 - a*exp(-j*w))); % exact DTFT
subplot(2,1,1)
  plot(w,fe)
  title('Exact spectrum');
  xlabel('\Omega'); ylabel('Magnitude');
   wh = 0.54 - 0.46*cos(2*pi*n'/(N-1)) %Hamming window
fh=abs((a.^n'.*wh)'*exp(-j*n'*w));
fhs = sum(fh,1); %sums columns of N x length(w) matrix fh
subplot(2,1,2)
   plot(w,fhs)
   title('Spectrum using Hamming window');
  xlabel('\Omega'); ylabel('Magnitude');
```

The exact spectrum and the spectrum obtained using the Hamming window are shown in Figure 6.6.

Practice Problem 6.14 Repeat Example 6.14 using the Blackman window.

Answer: See Figure 6.7.

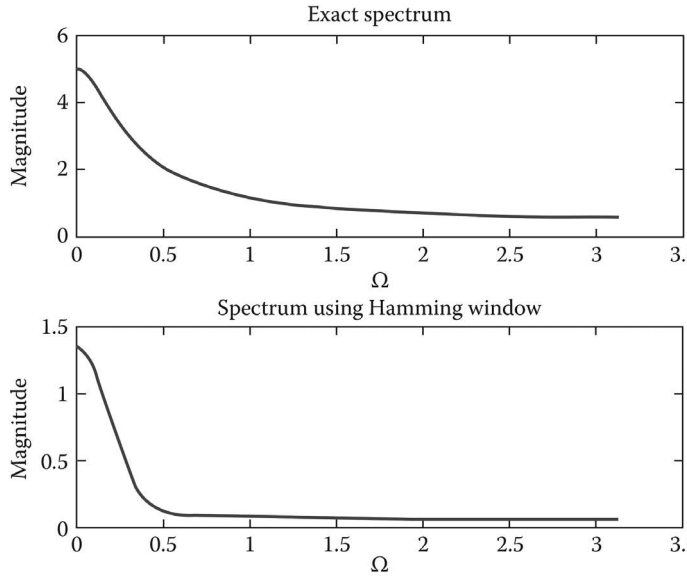


FIGURE 6.6 For Example 6.14.

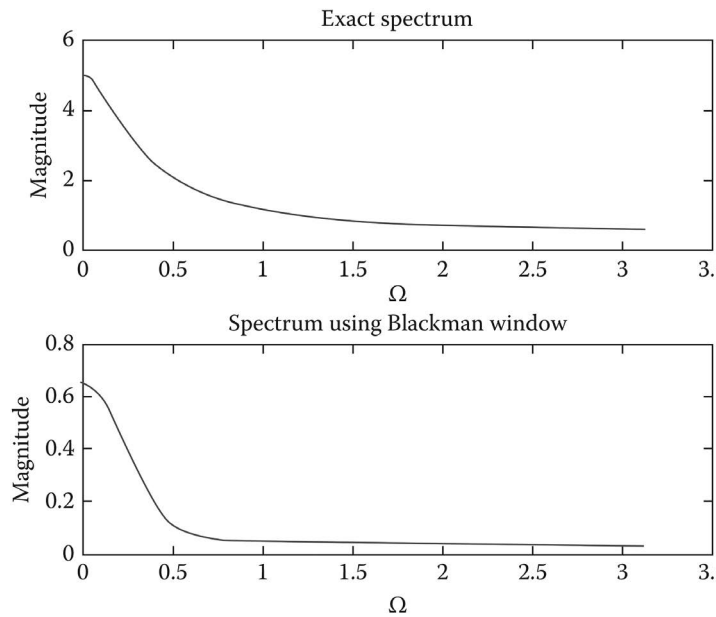


FIGURE 6.7 For Practice Example 6.14.

SUMMARY

1. The DTFT of a signal $x[n]$ is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

where the variable Ω has the units of radians.

2. The IDTFT of $X(\Omega)$ is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

3. The properties of DTFT are listed in Table 6.1, while Table 6.2 presents the transforms of some common transform pairs.
4. The DFT of a sequence $x[n]$ of length N is given by

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$$

The properties of DFT are listed in Table 6.3.

5. The IDFT is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}$$

6. The FFT is an algorithm designed for efficient and fast machine computation of the DFT.
7. The MATLAB commands **fft** and **ifft** can be used, respectively, to find the FFT and IFFT of a discrete-time sequence.
8. The two applications considered in this chapter are the touch-tone telephone and windowing technique. The touch-tone telephone uses signals of different frequencies to specify which button is being pushed. When a discrete-time signal has an infinite length, we can terminate it by multiplying it by a window function.

REVIEW QUESTIONS

6.1 The condition for signal $x[n]$ to have DTFT is that $x[n]$ is:
(a) integratable, (b) differentiable, (c) summable, (d) compressible

6.2 The DTFT of the discrete-time signal $x[n] = 1$, $n = 0, \pm1, \pm2,\ldots$, does not exist.
(a) True, (b) False

6.3 Which of these is not convolution:
(a) $x[n]*h[n]$, (b) $x[n]h[n]$, (c) $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$, (d) $\sum_{k=-\infty}^{\infty} h[k]x[n-k]$

6.4 If $X(\Omega)$ is the DTFT of $x[n]$, then the Fourier transform of $x[-n]$ is
(a) $X(\Omega)e^{-j\Omega}$, (b) $X(\Omega)e^{j\Omega}$ (c) $X(\Omega - 1)$, (d) $X(-\Omega)$

6.5 The DFT is discrete but not periodic.
(a) True, (b) False

6.6 Given the following discrete-time sequence $x[0] = 1$, $x[1] = -1$, $x[0] = 3$, $x[3] = 2$, the value of $X[0]$ is:
(a) $2 + j3$, (b) $2 - j4$, (c) 5, (d) none of the above

6.7 For a four-point computation of DFT, how many complex multiplications are involved?
(a) 4, (b) 8, (c) 16, (d) 32

6.8 An N-point computation of DFT can be done by FFT provided N is even.
(a) True, (b) False

6.9 The MATLAB command for finding IDFT is:
(a) fft, (b) ifft, (c) stem, (d) idft, (d) none of the above

6.10 Which of the following MATLAB commands is not used for the windowing technique?
(a) Hanning, (b) Hamming, (c) Blackman, (d) Bartlett

Answers: 6.1 c, 6.2 a, 6.3 b, 6.4 d, 6.5 b, 6.6 c, 6.7 c, 6.8 b, 6.9 b, 6.10 a.

PROBLEMS

Sections 6.2 and 6.3: Discrete-Time Fourier Transform and Properties

6.1 (a) Show that $X(\Omega)$ is periodic with period 2π; that is $X(\Omega + 2\pi) = X(\Omega)$.
(b) Specifically show that $X(\Omega) = \dfrac{1}{1 - ae^{-j\Omega}}$ is periodic.

6.2 Find the DTFT of the signal shown in Figure 6.8.

6.3 Determine the DTFT of the sequence $x[n]$ shown in Figure 6.9.

6.4 Compute the DTFT of the signals in Figure 6.10.

6.5 Obtain the DTFT of the sequence in Figure 6.11.

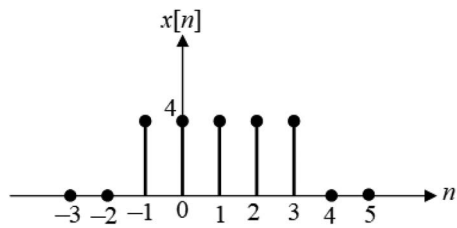


FIGURE 6.8 For Problem 6.2.

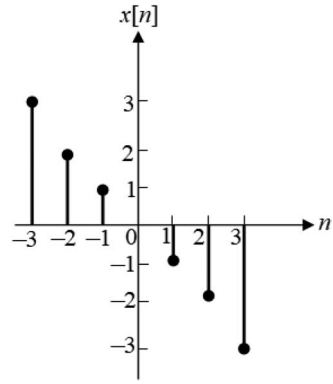


FIGURE 6.9 For Problem 6.3.

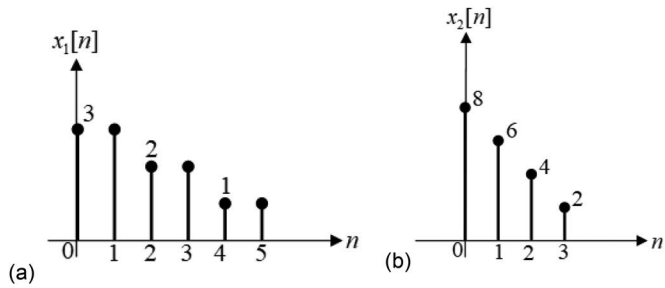


FIGURE 6.10 For Problems 6.4 and 6.32.

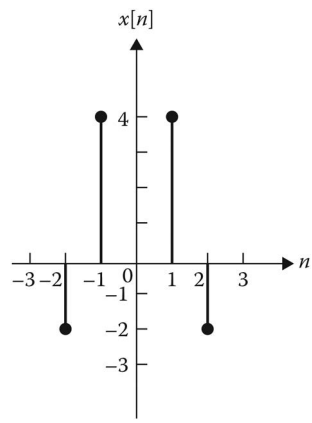


FIGURE 6.11 For Problem 6.5.

6.6 Find the DTFT of

$$x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Simplify your work.

6.7 Let the signal $y[n]$ be given by

$$y[n] = \begin{cases} 2, & n = -2 \\ -3, & n = -1 \\ 1, & n = 0 \\ 0, & n = 1 \\ 4, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Obtain the DTFT of $y[n]$.

6.8 Determine the DTFT $X(\Omega)$ for each of the following signals:

(a) $x[n] = 0.8^{|n|}$

(b) $x[n] = \delta[n + k] - \delta[n - k]$

(c) $x[n] = 0.4^n \cos 2nu[n]$

6.9 Find the DTFT of each of the following discrete-time signals:

(a) $(0.5)^n u[-n]$

(b) $\delta[n - 2] - \delta[n + 2]$

(c) $(0.8)^n u[n + 2]$

6.10 Obtain the DTFT of the following composite signals.

(a) $x[n] = (0.6)^n u[n] - \sin 2n$

(b) $y[n] = 2^n u[-n] + \cos 2\alpha n$

6.11 Use Tables 6.1 and 6.2 to find the DTFT of

$$x[n] = a^n u[n] - a^{-n} u[-n]$$

6.12 Given that

$$x[n] \leftrightarrow X(\omega) = \frac{1}{1 - ae^{-j\Omega}}$$

Find DTFT $Y(\Omega)$ of the following:

(a) $y[n] = nx[n2]$

(b) $z[n] = (n–)x[n]$

(c) $h[n] = (n - 1)x[n]$

6.13 The DTFT of a signal $x[n]$ is

$$X(\Omega) = \frac{2}{3 + e^{-j\Omega}}$$

Find the DTFT of the following signals.

(a) $y[n] = x[-n]$

(b) $z[n] = nx[n]$

(c) $w[n] = x[n] + x[n-1]$

(d) $v[n] = x[n]\cos(n\pi)$

6.14 Let $X(\Omega)$ be the DTFT of $x[n] = (0.8)^n u[n]$. Find the signal corresponding to the following spectra.

(a) $Y(\Omega) = X(-\Omega)$

(b) $Z(\Omega) = X(\Omega - \pi)$

(c) $W(\Omega) = jX'(\Omega)$

6.15 Obtain the IDTFT of each of the following Fourier transforms:

(a) $X(\Omega) = 10\sin(4\Omega)$

(b) $X(\Omega) = 2\cos^2\Omega$

6.16 Determine the signal $x[n]$ corresponding to each of the following Fourier transforms:

(a) $X(\Omega) = 1 + 3e^{-j2\Omega} - 2e^{j4\Omega} + e^{j5\Omega}$

(b) $X(\Omega) = \dfrac{e^{-j\Omega} - \dfrac{1}{2}}{1 - \dfrac{1}{2}e^{-j\Omega}}$

(c) $X(\Omega) = 3\pi\left[\delta(\Omega - 2) + \delta(\Omega + 2)\right]$

6.17 Obtain the inverse DTFT of

$$X(\Omega) = 3e^{j2\Omega} + 4 - 2e^{-j\Omega} + e^{-j3\Omega} + 5e^{-j5\Omega}$$

6.18 Determine the signal $x[n]$ with the following transform:

$$X(\Omega) = 10\cos 3\Omega + 6\sin 4\Omega$$

6.19 Obtain the IDTFT *x[n]* of the rectangular pulse spectrum $X(\Omega)$ shown in Figure 6.12.

6.20 Let $X(\Omega) = \begin{cases} 1, & -\pi < \Omega < \pi \\ 0, & \text{otherwise} \end{cases}$. Use this function to verify Parseval's relation.

6.21 Use eq. (6.8) to determine the IDTFT of each of the spectra in Figure 6.13.

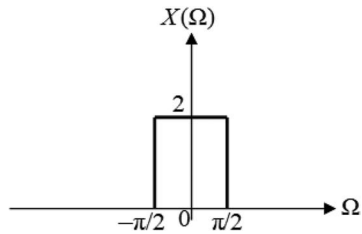


FIGURE 6.12 For Problem 6.19.

6.22 If $x[n] \leftrightarrow X(\Omega)$, show that

$$x[n+k]-x[n-k] \leftrightarrow 2jX(\Omega)\sin k\Omega$$

Hint: $x[n-m] \leftrightarrow e^{-j\Omega m}X(\Omega)$

6.23 Given that the DTFT pairs,

$$x[n] = a^n u[n] \leftrightarrow X(\omega) = \frac{1}{1-ae^{-j\Omega}}$$

(a) Show that

$$y[n] = na^n\, u[n] \leftrightarrow Y(\Omega) = \frac{ae^{-j\Omega}}{(1-a^{-j\Omega})^2}$$

(b) Show that $Y(\Omega)$ is periodic.

$$\text{Let} \quad x[n] = \left(\frac{1}{2}\right)^n u[n], \quad h[n] = \left(\frac{1}{4}\right)^n u[n]$$

6.24 Use DTFT to find the convolution

$$y[n] = x[n] * h[n]$$

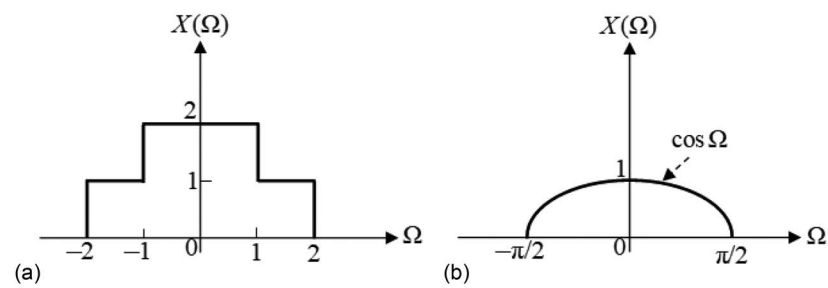


FIGURE 6.13 For Problem 6.21.

6.25 Find the convolution $y[n] = h[n]*x[n]$ of the following pairs of signals:

(a) $x[n] = \left(\frac{1}{4}\right)^n u[n], h[n] = 1$

(b) $x[n] = \left(\frac{1}{3}\right)^n u[n], h[n] = \delta[n] + \delta[n-1]$

(c) $x[n] = \left(\frac{1}{2}\right)^n u[n], h[n] = \left(\frac{1}{3}\right)^n u[n]$

Sections 6.4 and 6.5: Discrete Fourier Transform and Fast Fourier Transform

6.26 Find the DFT of the sequence $x[n] = a^n$.

6.27 Prove the following DFT properties

(a) $X[0] = \sum_{n=0}^{N-1} x[n]$

(b) $X[N/2] = \sum_{n=0}^{N-1} (-1)^n x[n]$

6.28 Obtain the DFT of the following discrete signal:

$$x[n] = \begin{cases} n, & 0 \le n \le 3 \\ 0, & \text{otherview} \end{cases}$$

6.29 Find the DFT of the following sequences:

(a) $x[n] = \{0, 1, 2, 3\}$

(b) $y[n] = \{1, 1, -1, -1, 1, 1, -1, -1\}$

6.30 Calculate the DFT of the following discrete-time signals:

(a) $x[0] = -1, x[1] = 1, x[2] = 0, x[3] = 2$

(b) $x[0] = 1, x[1] = 2, x[2] = 3, x[3] = -1$

6.31 Show that

(a) $x[n]\cos\left(\frac{2\pi km}{N}\right) \leftrightarrow \frac{1}{2}[X(m-k) + X(m+k)]$

(b) $x[n]\sin\left(\frac{2\pi km}{N}\right) \leftrightarrow \frac{1}{2j}[X(m-k) - X(m+k)]$

6.32 Determine the DFT of the signals in Figure 6.10.

6.33 The DFT of a signal $x[n]$ is

$$X(0) = 1, \quad X(1) = 1 + j2, \quad X(2) = 1 - j, \quad X(3) = 1 + j, \quad X(4) = 1 - j2$$

Compute $x[n]$.

6.34 The DFT of a signal $x[n]$ is

$$X(0)=2,\quad X(1)=j5,\quad X(2)=0,\quad X(3)=-j3$$

Find $x[n]$ and its energy.

6.35 Find the energy of a discrete-time sequence $x[n] = [2\ 1\ -1\ 4]$.

6.36 If the DFT of a signal $x[n]$ is

$$X[k]=[-1,\ 6+j,-3,\ 6-j]$$

Find $x[n]$.

6.37 (a) Show that the discrete-time Fourier transform $x[n] = a^n u[n]$ is

$$X(\Omega)=\frac{1}{1-ae^{-j\Omega}}$$

(b) Show that $X(\Omega)$ is periodic, i.e.

$$X(\Omega+2\pi)=X(\Omega)$$

6.38 Given that the DFT of a signal $x[n]$ is

$$X(0)=4,\quad X(1)=-1-j,\quad X(2)=-6,\quad X(3)=-1+j$$

Calculate $x[n]$.

6.39 Given that $X[k] = \text{DFT}\{x[n]\}$, show that

$$x[n]=\frac{1}{N}\left(DFT\left\{X^*[k]\right\}\right)^*$$

where * denotes complex conjugation. This shows that the same algorithm used for computing DFT can be used for computing IDFT.

6.40 How many multiplications are required in DFT and FFT when $N = 2048$?

Section 6.6: Computing with MATLAB

6.41 Use MATLAB to compute the FFT of the following signals. For each signal, plot $|X(k)|$.

(a) $x[n] = 1,\ 0 \le n \le 12$

(b) $x[n] = n,\ 0 \le n \le 10$

(c) $$x[n]=\begin{cases}1, & n=0\\ 1/n, & n=1,2,..,10\\ 0, & \text{otherwise}\end{cases}$$

(d) $x[n] = n(0.8)^n,\ 0 \le n \le 10$

6.42 In Example 6.3, the DTFT of the signal $x[n] = a^{|n|}$ is

$$X(\Omega) = \frac{1 - a^2}{1 - 2a\cos\Omega + a^2}$$

For $a = 0.75$ and $0 < \Omega < 2\pi$, plot $|X(\Omega)|$

6.43 Use MATLAB to find the DFT of the discrete signal

$$x[n] = \{1,\ 2,\ 0, -1, -2,\ 1,\ 5,\ 4\}$$

6.44 Consider the following 16-point sequence:

$$x[n] = \begin{cases} 1, & n = 0,1,\ldots 16 \\ 0, & \text{otherwise} \end{cases}$$

Using MATLAB, plot the absolute value of the DFT of $x[n]$.

Section 6.6: Applications

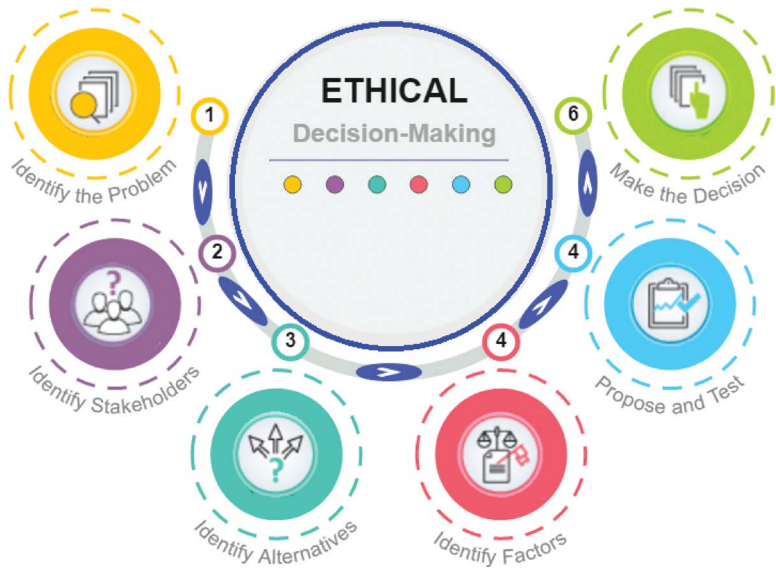
6.45 Use MATLAB to plot Blackman windowing for $N = 15$.

6.46 Modify the window analysis program of Example 6.14 by replacing the Hamming window with the Hanning window.

7 z-Transform

Let him who would enjoy a good future waste none of his present.

Roger Babson



CODES OF ETHICS

Engineering is a profession that makes significant contributions to the economic and social well-being of people all over the world. As members of this important profession, engineers are expected to exhibit the highest standards of honesty and integrity. Unfortunately, the engineering curriculum is so crowded that there is no room for a course on ethics in most schools. The Institute of Electrical and Electronics Engineers (or IEEE) Code of Ethics is presented here to acquaint students with ethical behavior in engineering professions.

We, the members of the IEEE, in recognition of the importance of our technologies in affecting the quality of life throughout the world, and in accepting a personal obligation to our profession, its members and the communities we serve, do hereby commit ourselves to the highest ethical and professional conduct and agree:

1. to accept responsibility in making engineering decisions consistent with the safety, health and welfare of the public, and to disclose promptly factors that might endanger the public or the environment;
2. to avoid real or perceived conflicts of interest whenever possible, and to disclose them to affected parties when they do exist;

DOI: 10.1201/9781003490869-7

3. to be honest and realistic in stating claims or estimates based on available data;
4. to reject bribery in all its forms;
5. to improve the understanding of technology, its appropriate application, and potential consequences;
6. to maintain and improve our technical competence and to undertake technological tasks for others only if qualified by training or experience, or after full disclosure of pertinent limitations;
7. to seek, accept, and offer honest criticism of technical work, to acknowledge and correct errors, and to credit properly the contributions of others;
8. to treat fairly all persons regardless of such factors as race, religion, gender, disability, age, or national origin;
9. to avoid injuring others, their property, reputation, or employment by false or malicious action;
10. to assist colleagues and co-workers in their professional development and to support them in following this code of ethics.

Courtesy of IEEE

7.1 INTRODUCTION

The z-transform is the discrete-time counterpart of the Laplace transform. It puts at our disposal a compact tool for describing a broad variety of systems and their properties.

Just as the Laplace transform is useful in handling signals which do not have a Fourier transform, the z-transform enables us to treat discrete-time signals that do not have discrete-time Fourier transforms (DTFTs). (For example, the unit sequence $u[n]$ does not have a DTFT.) Also, just as the Laplace transform converts integro-differential equations into algebraic equations, the z-transform converts differential equations into algebraic equations, which are easier to manipulate and solve. Therefore, techniques in this chapter parallel the techniques used in Chapter 3 on the Laplace transform. Although the properties of the z-transform are similar to those of the Laplace transform, there are some differences. Like the Laplace transform, the z-transform is applicable to systems with initial conditions.

We begin by defining the z-transform and studying its important properties. The z-transforms of some common discrete-time signals are derived. We present methods for finding the inverse z-transform. Two important applications of the z-transform are discussed. Finally, we consider using MATLAB® to find the z-transform and its inverse.

7.2 DEFINITION OF THE Z-TRANSFORM

The z-transform is the generalization of the DTFT. We recall that from Chapter 6 that the DTFT of a signal $x[n]$ is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{7.1}$$

Inserting a factor ρ^{-n} in eq. (7.1) leads to

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\rho^{-n}e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n](\rho e^{j\Omega})^{-n} \tag{7.2}$$

If we let $z = \rho e^{j\Omega}$. Then, eq. (7.2) becomes

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{7.3}$$

This is the two-sided (or bilateral) z-transform. The one-sided (or unilateral) z- transform is

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{7.4}$$

Only the one-sided z-transform will be discussed in this chapter. $X(z)$ and $x[n]$ constitute a z-transform pair.

$$x[n] \overset{Z}{\leftrightarrow} X(z) \tag{7.5}$$

Note the following:

1. From eq. (7.4), we notice that the z-transform of $x[n]$ is a power series of z^{-1} whose coefficients are $x[n]$, i.e.,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots \tag{7.6}$$

   In this power series, z^n can be interpreted as indicating the nth sampling instant.
2. We also notice that the sum in eq. (7.4) may not converge for all values of z. Each z-transform is associated with a region in the complex plane for which the transform exists. We will discuss this further in the next section.
3. Generally, z is a complex variable with $z = \Sigma + j\Omega$ just as the Laplace transform variable $s = \sigma + j\omega$. z as a complex number $z = \rho e^{j\Omega}$ is illustrated in Figure 7.1.
4. In our development of z-transforms, we will be using the formula for the geometric series

$$1 + a + a^2 + a^3 + \cdots = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \tag{7.7}$$

   which is convergent if and only if $|a| < 1$.

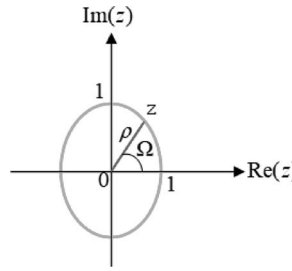


FIGURE 7.1 The unit circle in the z-plane.

Example 7.1

Let $x[n]$ by defined by

$$x[n] = \begin{cases} 1, & n = 1 \\ 3, & n = 4 \\ 0, & \text{otherwise} \end{cases}$$

Find the z-transform

SOLUTION

Using eq. (7.5),

$$X(z) = 1z^{-1} + 3z^{-4} = \frac{1}{z} + \frac{3}{z^4} = \frac{z^3 + 3}{z^4}$$

Practice Problem 7.1 Find the z-transform of the discrete-time signal

$$x[n] = \begin{cases} 2, & n = 11 \\ 4, & n = 42 \\ 0, & \text{otherwise} \end{cases}$$

Answer: $\dfrac{4 + 2z^{31}}{z^{42}}$.

Example 7.2

Determine the z-transform of the signal in Figure 7.2.

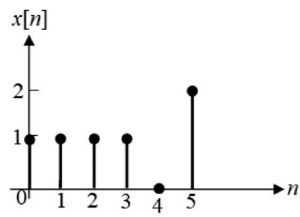


FIGURE 7.2 For Example 7.2.

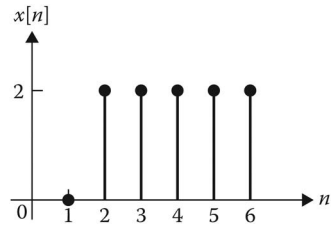


FIGURE 7.3 For Practice Problem 7.2.

SOLUTION

Using eq. (7.6),

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots$$
$$= 1 + (1)z^{-1} + (1)z^{-2} + (1)z^{-3} + (0)z^{-4} + (2)z^{-5} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{2}{z^5}$$
$$= \frac{z^5 + z^4 + z^3 + z^2 + 2}{z^5}$$

Practice Problem 7.2 Find the z-transform of the signal in Figure 7.3.

Answer: $\dfrac{2(1 + z + z^2 + z^3 + z^4)}{z^6}$

Example 7.3

Find the z-transform of the impulse sequence $\delta[n - m]$, $m \geq 0$.

SOLUTION

The given sequence can be written as

$$\delta[n-m] = \begin{cases} 1, & n = m \\ 0, & \text{otherwise} \end{cases}$$

that is, the sequence is zero for all n except when $n = m$. Hence the z-transform is

$$\mathcal{Z}\{\delta[n-m]\} = \sum_{n=0}^{\infty} \delta[n-m]z^{-n} = z^{-m} \tag{7.3.1}$$

Notice that if $m = 0$,

$$\mathcal{Z}\{\delta[n]\} = 1 \tag{7.3.2}$$

Practice Problem 7.3 Find the z-transform of the unit-sample sequence.

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Answer: 1.

7.3 REGION OF CONVERGENCE

Just like the Laplace transform, the z-transform $X(z)$ converges only for certain range of z known as the *region of convergence* (ROC). In order for the z-transform to be complete, its ROC must be specified. The ROC specifies the region where the z-transform converges. The ROC may be within a circle, such as $|z| < |a|$, as shown typically in Figure 7.4(a); outside a circle, such as $|z| > |a|$, as typically shown in Figure 7.4(b); or within an annulus, such as $|a| < |z| < |b|$, as shown typically in Figure 7.4(c). A ROC does not include a pole because the z-transform does not converge at a pole.

To illustrate how to find the ROC, let us consider the following examples.

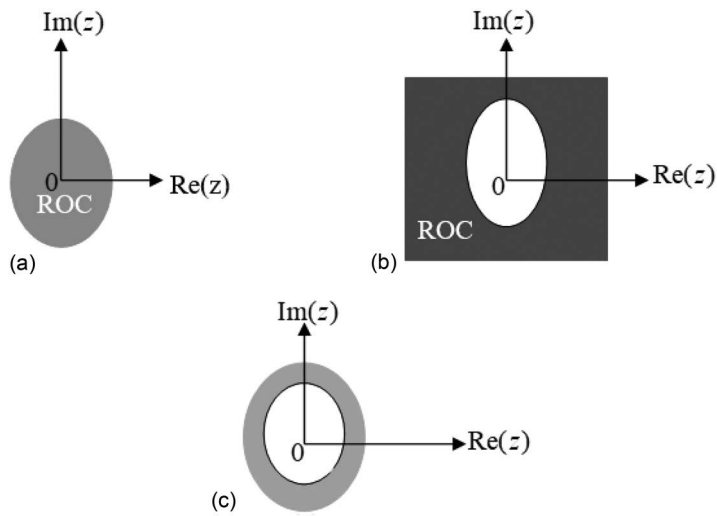


FIGURE 7.4 The regions of convergence (ROCs) of the z-transforms of various sequences.

Example 7.4

Determine the z-transform of an exponential discrete-time signal represented by $x[n] = a^n u[n]$.

SOLUTION

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots$$

This is a geometric series which has the form

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad |q| < 1$$

In our case, $q = a/z$ so that

$$X(z) = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

The region of convergence is $\left|\frac{a}{z}\right| < 1$ or $|z| > |a|$. This is similar to the ROC in Figure 7.2(b).

Practice Problem 7.4 Obtain the z-transform of the discrete-time unit step signal $x[n] = u[n]$. Specify the ROC.

Answer: $\frac{z}{z-1}, \quad |z| > 1.$

Example 7.5

Find the z-transform of the following composite signal and determine its region of convergence.

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 4\left(-\frac{1}{3}\right)^{2n} u[n]$$

SOLUTION

$x[n]$ consists of two signals. We may find the z-transform of each separately and add them.

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

From Example 7.4,

$$X_1(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > \frac{1}{2} \tag{7.51}$$

Similarly,

$$x_2[n] = 4\left(-\frac{1}{3}\right)^{2n} u[n] = 4\left(\frac{1}{9}\right)^{n} u[n]$$

Using the result from Example 7.3 again,

$$X_2(z) = \frac{4z}{z - \frac{1}{9}}, \quad |z| > \frac{1}{9} \tag{7.5.2}$$

Combining eqs. (7.5.1) and (7.5.2) gives

$$X(z) = X_1(z) + X_2(z) = \frac{z}{z - \frac{1}{2}} + \frac{4z}{z - \frac{1}{9}} = \frac{2z}{2z - 1} + \frac{36z}{9z - 1}$$

To determine the ROC, we realize that $X(z)$ exists if and only if both $X_1(z)$ and $X_2(z)$ exist. Hence the region of converge is $|z| > ½$, which covers the ROC of both $X_1(z)$ and $X_2(z)$.

Practice Problem 7.5 Determine the z-transform and the ROC for

$$y[n] = 4\left(\frac{1}{3}\right)^{n} u[n] - 5\left(\frac{2}{3}\right)^{n} u[n]$$

Answer: $Y(z) = \dfrac{12z}{3z - 1} - \dfrac{15z}{3z - 2}, |z| > \dfrac{2}{3}.$

7.4 PROPERTIES OF THE z-TRANSFORM

We now consider some basic properties of the z-transform. We use those properties to derive the z-transform of some signals and develop a z-transform table. We will first state each property, prove it, and then illustrate it with examples.

7.4.1 Linearity

Let $x[n]$ and $y[n]$ be two discrete-time signals with transforms $X(z)$ and $Y(z)$, respectively. The z-transform of their linear combination is

$$\mathcal{Z}\{ax[n]+by[n]\} = aX(z)+bY(z) \tag{7.8}$$

where a and b are constants.

For example, from Example 7.4 and Practice Problem 7.4, we know that

$$a^n u[n] \leftrightarrow \frac{z}{z-a}, \quad |z|>|a| \quad \text{and} \quad u[n] \leftrightarrow \frac{z}{z-1}, \quad |z|>1 \tag{7.9}$$

By the linearity property,

$$2a^n u[n]+3u[n] \leftrightarrow \frac{2z}{z-a}+\frac{3z}{z-1}, \quad |z|>|a| \quad \text{and} \quad |z|>1 \tag{7.10}$$

7.4.2 Time Shifting

This property states that if $X(z)$ is the z-transform of $x[n]$, then

$$x[n-m] \leftrightarrow z^{-m}X(z) \tag{7.11}$$

assuming zero initial conditions. To prove this, we use eq. (7.5).

$$\mathcal{Z}\{x[n-m]\} = \sum_{n=0}^{\infty} x[n-m]z^{-n}$$

By changing variables $k = n - m$, we get

$$\mathcal{Z}\{x[n-m]\} = \sum_{k=0}^{\infty} x[k]z^{-(m+k)} = z^{-m}\sum_{k=0}^{\infty} x[k]z^{-k} = z^{-m}X(z) \tag{7.12}$$

This assumes that the signal is causal ($x[n] = 0$ for $n < 0$).

For example, when $m = 1$ or –1, we have

$$\begin{aligned} x[n-1] &\leftrightarrow z^{-1}X(z) \\ x[n+1] &\leftrightarrow zX(z) \end{aligned} \tag{7.13}$$

For nonzero initial conditions, the time-delay property becomes

$$x[n-m] \leftrightarrow z^{-m}X(z) + z^{-m+1}x[-1] + \cdots + z^{-1}x[-m+1] + x[-m] \tag{7.14}$$

7.4.3 Frequency Scaling

If $X(z)$ is the z-transform of $x[n]$, then

$$a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right) \tag{7.15}$$

This implies that multiplication by a^n in the time domain corresponds to scaling in the frequency or z domain.

By definition, the z-transform of the sequence $a^n x[n]$ is

$$\begin{aligned} \mathcal{Z}\left\{a^n x[n]\right\} &= \sum_{n=0}^{\infty} a^n x[n] z^{-n} = \sum_{n=0}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} \\ &= X\left(\frac{z}{a}\right) \end{aligned} \tag{7.16}$$

This property can be used to find the z-transform of signals multiplied by an exponential signal. For example:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0} z) \tag{7.17}$$

We know that

$$\cos(\Omega n)u[n] = \frac{1}{2}\left(e^{j\Omega n} + e^{-j\Omega n}\right)u[n] \tag{7.18a}$$

and that

$$u[n] \leftrightarrow \frac{z}{z-1} \tag{7.18b}$$

$$e^{j\Omega n}\, u[n] \leftrightarrow \frac{ze^{j\Omega}}{ze^{j\Omega} - 1} = \frac{z}{z - e^{-j\Omega}} \tag{7.18c}$$

Similarly,

$$e^{-j\Omega n}\, u[n] \leftrightarrow \frac{z}{z - e^{j\Omega}} \tag{7.19}$$

Combining eqs. (7.17) and (7.18) gives

$$\mathcal{Z}\left\{\cos \Omega n u[n]\right\} = \frac{1}{2}\left[\frac{z}{z-e^{-j\Omega}} + \frac{z}{z-e^{j\Omega}}\right] = \frac{z(z-\cos\Omega)}{z^2 - 2z\cos\Omega + 1} \tag{7.20a}$$

7.4.4 Time Reversal

If $x[n] \leftrightarrow X(z)$,

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \tag{7.20b}$$

To prove this, we use eq. (7.5)

$$\mathcal{Z}\{x[-n]\} = \sum_{n=0}^{\infty} x[-n]z^{-n}$$

Replacing $-n$ with k,

$$\mathcal{Z}\{x[-n]\} = \sum x[k]z^{k} = \sum x[k]\left(\frac{1}{z}\right)^{-k} = X\left(\frac{1}{z}\right) \tag{7.21}$$

For example, from eq. (7.17),

$$u[n] \leftrightarrow \frac{z}{z-1}$$

Applying eq. (7.20),

$$u[-n] \leftrightarrow \frac{1/z}{1/z-1} = \frac{1}{1-z} \tag{7.22}$$

7.4.5 Modulation

The modulation property is very important in communication theory and digital signal processing. If $x[n] \leftrightarrow X(z)$, then

$$(\cos \Omega n)x[n] \leftrightarrow \frac{1}{2}\left[X(e^{j\Omega}z) + X(e^{-j\Omega}z)\right] \tag{7.23}$$

$$(\sin \Omega n)x[n] \leftrightarrow \frac{j}{2}\left[X(e^{j\Omega}z) - X(e^{-j\Omega}z)\right] \tag{7.24}$$

To prove eqs. (7.23) and (7.24), we use Euler's identity:

$$\cos \Omega n x[n] = \frac{1}{2}\left(e^{-j\Omega n}\, x[n] + e^{j\Omega n}\, x[n]\right) \tag{7.25}$$

$$\sin \Omega n x[n] = \frac{j}{2}\left(e^{-j\Omega n}\, x[n] - e^{j\Omega n}\, x[n]\right) \tag{7.26}$$

From eq. (7.16),

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(e^{-j\Omega_0} z) \tag{7.27a}$$

$$e^{-j\Omega_0 n} x[n] \leftrightarrow X(e^{j\Omega_0} z) \tag{7.27b}$$

Applying eq. (7.27) to eqs. (7.25) and (7.26) produces eqs. (7.23) and (7.24). We can use this property to find the z-transforms of cosine and sine signals. We already got the z-transform of the cosine in eq. (7.20).

7.4.6 Accumulation

Let $y[n]$ be the summation or accumulation of the discrete-time signal $x[n]$, with $x[n] = 0$ for $n = -1, -2, -3, \ldots$. Then

$$y[n] = \sum_{k=0}^{n} x[k] \tag{7.28}$$

The accumulation property states that

$$\sum_{k=0}^{n} x[k] \leftrightarrow \frac{z}{z-1} X(z) \tag{7.29}$$

It states that the z-transform of the sum of $x[n]$ is equal to the $z/(z-1)$ times the z-transform of $x[n]$.

To prove eq. (7.29), we write

$$y[n] = \sum_{k=0}^{n-1} x[k] + x[n] \tag{7.30}$$

This may be written as

$$y[n] = y[n-1] + x[n] \tag{7.31}$$

Taking the z-transform of both sides and applying the time-shifting property, we obtain

$$Y(z) = \frac{1}{z} Y(z) + X(z) \tag{7.32}$$

Solving for $Y(z)$ yields

$$Y(z) = \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z) \tag{7.33}$$

7.4.7 Convolution

This property plays a very useful role in linear system theory. It states that if

$$x[n] \leftrightarrow X(z) \quad \text{and} \quad h[n] \leftrightarrow H(z) \tag{7.34}$$

then their time convolution in the time domain is equivalent to their product in the frequency or z domain; i.e.,

$$y[n] = x[n] * h[n] \leftrightarrow Y(z) = X(z)H(z) \tag{7.35}$$

Since the property applies to causal as well as noncausal signals, we prove it to the general case of noncausal signals, where the convolution sum ranges from $-\infty$ to ∞. By definition,

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x[n] * h[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[n] h[n-k] \right) z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k] z^{-n} \right) \end{aligned} \tag{7.36}$$

We note that the term in brackets is the z-transform of the shifted signal $h[n - k]$. By applying the time-shifting property, we get

$$Y(z) = \sum_{k=-\infty}^{\infty} x[k] \left(z^{-k} H(z) \right) = \left(\sum_{k=-\infty}^{\infty} x[k] z^{-k} \right) H(z) = X(z)H(z) \tag{7.37}$$

7.4.8 Initial and Final Values

The initial $x[0]$ and the final value $x[\infty]$ are given by the initial and value theorems, which are stated as

$$x[0] = \lim_{z \to \infty} X(z) \tag{7.38}$$

$$x[\infty] = \lim_{z \to 1} (1 - z^{-1}) X(z) \tag{7.39}$$

assuming that $x[\infty]$ exists.

To prove eq. (7.38), we write eq. (7.5) explicitly as

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[k]z^{-k} + \cdots \tag{7.40}$$

As $z \to \infty$, the terms $z^{-k} \to 0$ for all $k > 0$. Hence, eq. (7.40) becomes

$$\lim_{z\to\infty} X(z) = x[0]$$

To prove eq. (7.39), we apply the time-shifting property:

$$\mathcal{Z}\{x[n] - x[n-1]\} = (1 - z^{-1})X(z) \tag{7.41}$$

The left-hand side of eq. (7.41) can be written as

$$\sum_{n=0}^{\infty} \{x[n] - x[n-1]\} z^{-n} = \lim_{N\to\infty} \sum_{n=0}^{N} \{x[n] - x[n-1]\} z^{-n} \tag{7.42}$$

If we take the limit as $z \to 1$, combining eqs. (7.41) and (7.42) produces

$$\begin{aligned} \lim_{z\to 1}(1 - z^{-1})X(z) &= \lim_{N\to\infty} \sum_{n=0}^{\infty} \{x[n] - x[n-1]\} \\ &= \lim_{N\to\infty} x(N) = x(\infty) \end{aligned} \tag{7.43}$$

The initial value theorem can be used to check the z-transform of a given signal. The final value theorem gives the correct result only when $x[\infty]$ exists. It can be shown that if there are poles on or outside the unit circle, except for a single pole at $z = 1$, the final value theorem does not apply.

For example, let us find the initial and value values of a unit step function, $u[n]$. We know that $u[0] = 1 = u[\infty]$. By the initial value theorem,

$$u[0] = \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \left(\frac{z}{z-1} \right) = 1 \tag{7.44}$$

By the final value theorem,

$$x[\infty] = \lim_{z\to 1}(1 - z^{-1})X(z) = \lim_{z\to 1} \left(\frac{z-1}{z} \frac{z}{z-1} \right) = 1 \tag{7.45}$$

The theorems confirm what we know already.

TABLE 7.1
Properties of the z-Transforms

Property	$x[n]$	$X(z)$
1. Linearity	$ax[n]+by[n]$	$aX(z)+bY(z)$
2. Time-shifting	$x[n-m]$	$z^{-m}X(z)+z^{-m+1}x[-1]+\cdots+z^{-1}x[-m+1]+x[-m]$
3. Frequency scaling	$a^n x[n]$	$X\left(\frac{z}{a}\right)$
4. Time reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$
5. Multiplication by n	$nx[n]$	$-z\frac{d}{dz}X(z)$
6. Multiplication by n^2	$n^n x[n]$	$z\frac{d}{dz}X(z)+z^2\frac{d^2}{dz^2}X(z)$
7. Modulation:		
Multiplication by $e^{j\Omega_0 n}$	$e^{j\Omega_0 n}x[n]$	$X(e^{-j\Omega_0}z)$
Multiplication by $\cos\Omega n$	$(\cos\Omega n)x[n]$	$\frac{1}{2}\left[X(e^{j\Omega}z)+X(e^{-j\Omega}z)\right]$
Multiplication by $\sin\Omega n$	$(\sin\Omega n)x[n]$	$\frac{j}{2}\left[X(e^{j\Omega}z)-X(e^{-j\Omega}z)\right]$
8. Accumulation	$\sum_{k=0}^{n}x[k]$	$\frac{z}{z-1}X(z)$
9. Convolution	$x[n]*h[n]$	$X(z)H(z)$
10. Initial value	$x[0]=\lim_{z\to\infty}X(z)$	
11. Final value	$x[\infty]=\lim_{z\to 1}(1-z^{-1})X(z)$	

These properties of the z-transform are listed in Table 7.1, while Table 7.2 provides a collection of the z-transforms of common signals.

Example 7.6

Find the z-transform of the pulse function defined by

$$p[n]=\begin{cases}1, & n=0,1,2,\ldots,m-1\\ 0, & \text{otherwise}\end{cases}$$

SOLUTION

We can express $p[n]$ in terms of the unit-step function.

$$p[n]=u[n]-u[n-m]$$

TABLE 7.2
Some Common z-Transform Pairs

$x[n]$	$X(z)$	ROC
1. $\delta[n]$	1	all z
2. $\delta[n-m]$	$1/z^m$	$z \neq 0$
3. $u[n]$	$\frac{z}{z-1}$	$\|z\| > 1$
4. $a^n u[n]$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
5. $nu[n]$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
6. $(n+1)u[n]$	$\frac{z^2}{(z-1)^2}$	$\|z\| > 1$
7. $n^2 u[n]$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
8. $na^n u[n]$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9. $(n+1)a^n$	$\frac{z^2}{(z-a)^2}$	$\|z\| > \|a\|$
10. $n^2 a^n u[n]$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > \|a\|$
11. $\exp[-anT]$	$\frac{z}{z-\exp[-aT]}$	$\|z\| > e^{-aT}$
12. $\cos\Omega nu[n]$	$\frac{z^2 - z\cos\Omega}{z^2 - 2z\cos\Omega + 1}$	$\|z\| > 1$
13. $\sin\Omega nu[n]$	$\frac{z\sin\Omega}{z^2 - 2z\cos\Omega + 1}$	$\|z\| > 1$
14. $a^n(\cos\Omega n)u[n]$	$\frac{z^2 - za\cos\Omega}{z^2 - 2za\cos\Omega + a^2}$	$\|z\| > \|a\|$
15. $a^n(\sin\Omega n)u[n]$	$\frac{za\sin\Omega}{z^2 - 2za\cos\Omega + a^2}$	$\|z\| > \|a\|$

Taking the z-transform of each term and applying the time-shifting property, we get

$$P(z) = \frac{z}{z-1} - z^{-m}\frac{z}{z-1} = \frac{z(1-z^{-m})}{z-1}$$

Practice Problem 7.6 Let $p[n]$ be as defined in Example 7.6. Derive the z-transform of

$$q[n] = a^n p[n]$$

Answer: $Q(z) = \dfrac{z(1-a^m z^{-m})}{z-a}$

Example 7.7

Derive the z-transform of $y[n] = na^n u[n]$.

SOLUTION

We let $x[n] = a^n u[n]$. From entry 4 in Table 7.2,

$$X(z) = \frac{z}{z-a}$$

We now apply the multiplication-by-n property, which states that

$$\mathcal{Z}\{nx[n]\} = -z\frac{d}{dz}X(z)$$

But

$$z\frac{d}{dz}X(z) = \frac{-az}{(z-a)^2}$$

Thus,

$$Y(z) = Z\{na^n u[n]\} = \frac{az}{(z-a)^2}$$

Practice Problem 7.7 Determine the z-transform of $y[n] = n^2 a^n u[n]$.

Answer: $\dfrac{az(z+a)}{(z-a)^3}$.

Example 7.8

Use the z-transform to find the convolution of the following two sequences:

$$x[n] = [1, -2, 1, 4] \quad \text{and} \quad h[n] = [2, 0, 1, -3, 1]$$

SOLUTION

We use eq. (7.5) to find the z-transform of $x[n]$ and $h[n]$.

$$X(z) = 1 - 2z^{-1} + z^{-2} + 4z^{-3}$$

$$H(z) = 2 + 0z^{-1} + z^{-2} - 3z^{-3} + z^{-4}$$

Applying the convolution property gives

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= 2 + 0z^{-1} + z^{-2} - 3z^{-3} + z^{-4} \\ &\quad -4z^{-1} + 0z^{-2} - 2z^{-3} + 6z^{-4} - 2z^{-5} \\ &\quad +2z^{-2} + 0z^{3} + z^{-4} - 3z^{-5} + z^{-6} \\ &\quad +8z^{-3} + 0z^{-4} + 4z^{-5} - 12z^{-6} + 4z^{-7} \\ &= 2 - 4z^{-1} + 3z^{-2} + 3z^{-3} + 8z^{-4} - z^{-5} - 11z^{-6} + 4z^{-7} \end{aligned}$$

From this, we obtain $y[n]$ as

$$y[n] = [2, -4, 3, 3, 8, -1, -11, 4].$$

Practice Problem 7.8 Use the z-transform to find the convolution of these sequences:

$$x[n] = [1, 1, 1, 1] \quad \text{and} \quad h[n] = [0, 1, 0.5, 1, 1.5]$$

Answer: [0, 1, 1.5, 2.5, 4, 3, 2.5, 1.5].

Example 7.9

Let the z-transform of a signal be

$$X(z) = \frac{z - 0.75}{z(z-1)(z+0.75)}$$

Find the initial and final values.

SOLUTION

Applying the initial and final value theorems, we obtain

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{z-0.75}{z(z-1)(z+0.75)}$$

$$= \lim_{z\to\infty} \frac{1/z^2 - 0.75/z^3}{1(1-1/z)(1+0.75/z)} = 0$$

$$x[\infty] = \lim_{z\to 1} \frac{z-1}{z} X(z) = \lim_{z\to 1}\left(\frac{z-1}{z} \frac{z-0.75}{z(z-1)(z+0.75)} \right)$$

$$= \lim_{z\to 1}\left(\frac{z-0.75}{z^2(z+0.75)} \right) = \frac{1-0.75}{1^2(1+0.75)} = \frac{1}{7} = 0.1429$$

Practice Problem 7.9 Obtain the initial and final values of the signal whose z-transform is given by

$$X(z) = \frac{1+2z^{-1}}{1-1.6z^{-1}+0.6z^{-1}}$$

Answer: 1, 7.5.

Example 7.10

Find the transfer function $H(z) = Y(z)/X(z)$ for the causal system described by the following difference equation

$$y[n] - 3y[n-1] + 4y[n-2] = x[n] + x[n-3]$$

SOLUTION

Applying linearity and time-shifting properties, we have

$$Y(z) - 3z^{-1}\,Y(z) + 4z^{-2}\,Y(z) = X(z) + z^{-3}\,X(z)$$

or

$$Y(z)\left\{1 - 3z^{-1} + 4z^{-2}\right\} = X(z)\left\{1 + z^{-3}\right\}$$

Thus,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-3}}{1-3z^{-1}+4z^{-2}} = \frac{z^3+1}{z^3-3z^2+4z}$$

Practice Problem 7.10 Given the difference equation

$$y[n+3] + 2y[n+1] + y[n] = x[n+2] - x[n]$$

Find the transfer function $H(z)$.

Answer: $H(z) = \dfrac{z^2-1}{z^3+2z+1}$.

7.5 INVERSE z-TRANSFORM

The problem of finding the inverse z-transform is the problem of finding $x[n]$ for a given $X(z)$. The direct way to do this is to evaluate the contour integral:

$$x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z) z^{n-1}\, dz \tag{7.46}$$

where Γ is a closed contour that encloses the origin and includes all poles of $X(z)$. Γ is usually determined by the ROC of the summation in eq. (7.4). It encircles all singularities in the ROC. This direct method of finding the inverse transformation is computationally involved and will not be pursued further. We would rather use two other methods: long division expansion and partial fraction expansion.

7.5.1 Long Division Expansion

To find the inverse z-transform of $X(z)$, we expand $X(z) = N(z)/D(z)$ in a power series in z^{-1} by using long division. The values of $x[n]$ are taken as the coefficients of the series expansion. The long division becomes tedious if more than a few values are needed. This major deficiency limits the usefulness of the long division approach.

Example 7.11

Use long division expansion to find $x[n]$ given that

$$X(z) = \frac{z^2 - 2}{z^3 + z + 3}$$

SOLUTION

We carry out the long division as follows:

$$\begin{array}{r} z^{-1} + 0z^{-2} - 3z^{-3} + 3z^{-4} \\ z^3 + z + 3 \overline{)\, z^2 - 2} \\ \hline z^2 + 1 + 3z^{-1} \\ \hline -3 + 3z^{-1} \\ -3 - 3z^{-2} - 9z^{-3} \\ \hline 3z^{-1} + 3z^{-2} + 9z^{-3} \\ 3z^{-1} + 3z^{-3} + 9z^{4} \\ \hline 3z^{-2} + 6z^{-3} - 9z^{-4} \end{array}$$

Hence,

$$X(z) = z^{-1} - 3z^{-3} + 3z^{-4} + \cdots$$

We compare this with the basic definition of the z-transform in eq. (7.7), i.e.,

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots$$

Thus,

$$x[0] = 0, \quad x[1] = 1, \quad x[2] = 0, \quad x[3] = -3, \quad x[4] = 3, \ldots$$

Practice Problem 7.11 Using long division expansion, find the inverse transform of

$$X(z) = \frac{z}{z - 0.8}$$

Answer: $(0.8)^n u[n]$.

Example 7.12

Find the inverse z-transform of

$$X(z) = \log\left(\frac{z}{z-a}\right), \quad |z| > |a|$$

SOLUTION

This can be written as

$$X(z) = \log\left(\frac{1}{1 - az^{-1}}\right) = -\log(1 - az^{-1}) \tag{7.12.1}$$

The Maclaurin series expansion for $\log(1 - u)$ is given by

$$\log(1 - r) = -\sum_{n=1}^{\infty} \frac{1}{n} r^n \tag{7.12.2}$$

Applying eq. (7.12.2) in eq. (7.12.1) with $r = az^{-1}$ leads to

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n} a^n z^{-n} \tag{7.12.3}$$

We obtain $x[n]$ as the coefficients of the series in eq. (7.12.3).

$$x[n] = \frac{1}{n} a^n u[n-1]$$

Practice Problem 7.12 By long division, find $x[n]$ corresponding to

$$X(z) = \frac{z+1}{z+2}$$

Answer: $\delta[n] - \delta[n-1] + 2\delta[n-2] - 4\delta[n-3] + \cdots$

7.5.2 Partial Fraction Expansion

We use partial fraction expression to fins the inverse z-transform in the same way we used it to find the inverse Laplace transform in Section 3.4. In using partial fraction, a function $X(z)$ that does not appear in the z-transform in Table 7.2 is expressed as a sum of functions which are listed in the table. The inverse z-transform is then computed term by term using the appropriate entries in Table 7.2. We may also need to use the z-transform properties in Table 7.1. We will illustrate with examples for the cases of simple poles, repeated poles, and complex poles.

Example 7.13

Find the signal corresponding to

$$X(z) = \frac{4z}{(z+1)(z-0.8)}$$

SOLUTION

This example is on simple poles. If we expand $X(z)$ as

$$X(z) = \frac{4z}{(z+1)(z-0.8)} = \frac{A}{z+1} + \frac{B}{z-0.8}$$

we will find out that the terms $1/(z+1)$ and $1/(z-0.8)$ are not in Table 7.2. Thus, in order to get the inverse of $X(z)$, we need some additional manipulation. If we divide $X(z)$ by z, we get

$$\frac{X(z)}{z} = X_1(z) = \frac{4}{(z+1)(z-0.8)} = \frac{A}{z+1} + \frac{B}{z-0.8}$$

$$A = (z+1)X_1\big|_{z=-1} = \left.\frac{4}{z-0.8}\right|_{z=-1} = \frac{4}{-1.8} = -2.222$$

$$B = (z-0.8)X_1\big|_{z=0.8} = \left.\frac{4}{z+1}\right|_{z=0.8} = \frac{4}{1.8} = 2.222$$

Thus,

$$X(z) = \frac{20z}{z+1} + \frac{2.22z}{z-0.8}$$

From Table 7.2, we obtain the inverse as

$$u[n] = 20(-1)^n + 2.22(0.8)^n$$

Practice Problem 7.13 Find the inverse of

$$X(z) = \frac{4z-1}{(z-1)(z-2)}$$

Answer: $-0.5\delta[n] - 3u[n] + 3.5(2)^n u[n]$.

Example 7.14

Find the inverse z-transform of the function

$$X(z) = \frac{z+1}{(z-0.5)(z-1)^2}$$

SOLUTION

This example is on repeated poles. We let

$$X_1(z) = \frac{X(z)}{z} = \frac{z+1}{z(z-0.5)(z-1)^2} = \frac{A}{z} + \frac{B}{z-0.5} + \frac{C}{z-1} + \frac{D}{(z-1)^2} \qquad (7.14.1)$$

We obtain the expansion coefficients as

$$A = zX_1(z)\Big|_{z=0} = \frac{z+1}{(z-0.5)(z-1)^2}\Big|_{z=0} = \frac{1}{(-0.5)(-1)^2} = -2$$

$$B = (z-0.5)X_1(z)\Big|_{z=0.5} = \frac{z+1}{z(z-1)^2}\Big|_{z=0.5} = \frac{1.5}{0.5(-0.5)^2} = 12$$

$$D = (z-1)^2 X_1(z)\Big|_{z=1} = \frac{z+1}{z(z-0.5)}\Big|_{z=1} = \frac{2}{1(0.5)} = 4$$

To find C, we can select any appropriate value of z and substitute in eq. (7.14.1). If we choose $z = 2$, we obtain

$$X_1(z) = \frac{X(z)}{z} = \frac{z+1}{z(z-0.5)(z-1)^2} = \frac{A}{z} + \frac{B}{z-0.5} + \frac{C}{z-1} + \frac{D}{(z-1)^2}$$

$$\frac{2+1}{2(1.5)(1)} = \frac{-2}{2} + \frac{12}{1.5} + \frac{C}{1} + \frac{4}{1}$$

$$\text{Or } 1 = -1 + 8 + C + 4 \;\rightarrow\; C = -10$$

Thus,

$$X(z) = -2 + \frac{12z}{z-0.5} - \frac{10z}{z-1} + \frac{4z}{(z-1)^2}$$

Taking the inverse z-transform of each term,

$$x[n] = -2\delta[n] + \left\{12(0.5)^n - 10 + 4n\right\}u[n]$$

Practice Problem 7.14 Obtain the inverse z-transform for the function

$$X(z) = \frac{z-6}{(z-1)(z-2)^2}$$

Answer: $-1.5\delta[n] - \left\{5 - 4.5(2)^n + 2n(2)^n\right\}u[n]$.

Example 7.15

Find the signal corresponding to the z-transform

$$X(z) = \frac{2z^2 - 3z}{(z-1)(z^2 - 4z + 5)}$$

SOLUTION

This is an example on complex poles. Setting $z^2 - 4z + 5 = (z-2)^2 - 4 + 5 = (z-2)^2 - (j)^2 = (z-2-j)(z-2+j)$ reveals that there are complex poles at $2 \pm j$. We really do not need to know about these poles to find the inverse. We let

$$X_1(z) = \frac{X(z)}{z} = \frac{2z-3}{(z-1)(z^2-4z+5)} = \frac{A}{z-1} + \frac{Bz+C}{z^2-4z+5}$$

or

$$\frac{2z-3}{(z-1)(z^2-4z+5)} = \frac{A(z^2-4z+5) + (z-1)(Bz+C)}{(z-1)(z^2-4z+5)} \qquad (7.15.1)$$

Setting the numerators on both sides equal:

$$2z - 3 = A(z^2 - 4z + 5) + B(z^2 - z) + C(z-1)$$

Equating the coefficients:

$$z^2: 0 = A + B \text{ or } B = -A$$

$$z: 2 = -4A - B + C$$

$$\text{constant: } -3 = 5A - C$$

Solving these gives $A = -½$, $B = ½$, $C = ½$.
Thus,

$$X(z) = \frac{-0.5z}{z-1} + 0.5\frac{z^2+z}{z^2-4z+5} \tag{7.15.2}$$

The inverse of the first term is $-0.5u[n]$. We find the inverse of the second term by comparing with entries 15 and 16 in Table 7.2, namely,

$$a^n(\cos\Omega n)u[n] \leftrightarrow \frac{z^2 - za\cos\Omega}{z^2 - 2za\cos\Omega + a^2}$$

$$a^n(\sin\Omega n)u[n] \leftrightarrow \frac{za\sin\Omega}{z^2 - 2za\cos\Omega + a^2}$$

Set $z^2 - 4z + 5 = z^2 - 2az\cos\Omega + a^2$.

We notice that $a = \sqrt{5}$ and $2a\cos\Omega = 4 \rightarrow \cos\Omega = \frac{2}{\sqrt{5}}$ and consequently, $\sin\Omega = \frac{1}{\sqrt{5}}$ and $\Omega = 0.4636$. Thus,

$$X(z) = \frac{-0.5z}{z-1} + 0.5\frac{z^2-2z}{z^2-4z+5} + 0.5\frac{3z}{z^2-4z+5}$$

The inverse is

$$x[n] = 0.5\left\{-1 + \left(\sqrt{5}\right)^n \cos(0.4636n) + 3\left(\sqrt{5}\right)^n \sin(0.4636n)\right\}u[n]$$

Practice Problem 7.15 Obtain the inverse z-transform of

$$X(z) = \frac{2z}{z^2+4z+8}$$

Answer: $\left(\sqrt{8}\right)^n \sin\left(\frac{n\pi}{4}\right)u[n]$.

7.6 APPLICATIONS

The z-transform is fundamentally important to digital signal processing, digital communications, and linear control systems. In this section, we consider the use of the z-transform in the analysis of discrete-time linear systems. We will apply the z-transform to two areas: the linear difference equation and transfer function.

7.6.1 Linear Difference Equation

Most discrete-time systems of practical interest can be described by linear difference equations. We can use the z-transform to solve the difference equation for $n \geq 0$. To do this requires that we first take the z-transform of the two sides of the difference equation. Next, we solve algebraically the transformed difference equation. At this point, we can evaluate $Y(z)$ and invert it using Table 7.2 to determine $y[n]$.

To obtain the complete solution requires that we know the input signal $x[n]$. For Nth-order difference equations, we must also know the N initial conditions. Using the initial conditions, we obtain the z-transform of the delayed output signals as follows:

$$\begin{aligned}
\mathcal{Z}\{y[n-1]\} &= z^{-1}Y(z) + y[-1] \\
\mathcal{Z}\{y[n-2]\} &= z^{-2}Y(z) + z^{-1}y[-1] + y[-2] \\
\mathcal{Z}\{y[n-3]\} &= z^{-3}Y(z) + z^{-2}y[-1] + z^{-1}y[-2] + y[-3]
\end{aligned} \tag{7.47}$$

In general,

$$\mathcal{Z}\{y[n-m] = z^{-m}Y(z) + z^{-m}\sum_{n=1}^{m} y[-n]z^{n} \tag{7.48}$$

Similarly, the z-transforms of advanced output signals are:

$$\begin{aligned}
\mathcal{Z}\{y[n+1]\} &= zY(z) - zy[0] \\
\mathcal{Z}\{y[n+2]\} &= z^2Y(z) - z^2y[0] - zy[1] \\
\mathcal{Z}\{y[n+3]\} &= z^3Y(z) - z^3y[0] - z^2y[1] - zy[2]
\end{aligned} \tag{7.49}$$

In general,

$$\mathcal{Z}\{y[n+m] = z^{m}Y(z) - z^{m}\sum_{n=0}^{m-1} y[n]z^{-n} \tag{7.50}$$

We will illustrate this with an example.

Example 7.16

A second-order discrete-time system is characterized by the difference equation

$$y[n] + 0.2y[n-1] - 0.5y[n-2] = x[n] + x[n-1]$$

Find the complete solution for the initial conditions $y[-1] = 1$, $y[-2] = 2$ and an input $x[n] = u[n]$.

SOLUTION

Taking the z-transform of each term in the difference equation yields

$$Y(z)+0.2\left\{z^{-1}Y(z)+y[-1]\right\}-0.5\left\{z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\right\}=X(z)+\left\{z^{-1}X(z)+x[-1]\right\}$$

We should set $x[-1] = 0$ since $u[n] = 0$ for $n < 0$. Also,

$$X(z)=\frac{1}{1-z^{-1}}$$

Introducing the initial conditions for $y[-1]$ and $y[-2]$ leads to

$$Y(z)\left\{1+0.2z^{-1}-0.5z^{-2}\right\}+0.2-0.5z^{-1}-1=X(z)\left(1+z^{-1}\right)=\left(1+z^{-1}\right)/\left(1-z^{-1}\right)$$

$$Y(z)=\frac{0.8+0.5z^{-1}+(1+z^{-1})/(1-z^{-1})}{1+0.2z^{-1}-0.5z^{-2}}=\frac{0.8+0.5z^{-1}-0.8z^{-1}-0.5z^{-2}+1+z^{-1}}{(1-z^{-1})(1+0.2z^{-1}-0.5z^{-2})}$$

$$=\frac{1.8z^3+0.7z^2-0.5z}{(z-1)(z^2+0.2z-0.5)}$$

We use partial fraction to obtain $y[n]$.

$$Y_1(z)=\frac{Y(z)}{z}=\frac{1.8z^2+0.7z-0.5}{(z-1)(z+0.8141)(z-0.6141)}=\frac{A}{z-1}+\frac{B}{z+0.8141}+\frac{C}{z-0.6141}$$

$$A=(z-1)Y_1(z)\big|_{z=1}=\frac{1.8+0.7-0.5}{1.8141\times 0.3859}=2.8569$$

$$B=(z+0.8141)Y_1(z)\big|_{z=-0.8141}=\frac{1.193-0.5699-0.5}{(-1.8141)(-1.4282)}=0.0475$$

$$C=(z-0.6141)Y_1(z)\big|_{z=0.6141}=\frac{0.6788+0.4299-0.5}{(-0.3859)(1.4282)}=-1.1044$$

Thus,

$$Y(z)=\frac{2.8569z}{z-1}+\frac{0.0475z}{z+0.8141}-\frac{1.1044z}{z-0.6141}$$

By taking the inverse z-transform of each term, we finally obtain

$$y[n]=\left\{2.8569+0.0475(-0.8141)^n-1.1044(0.6141)^n\right\}u[n]$$

Practice Problem 7.16 Use the z-transform to solve the difference equation

$$y[n]+0.6y[n-1]=x[n]$$

where $x[n] = 4\delta[n-1]$ and $y[-1] = -2$.

Answer: $6.667\delta[n]-5.4667(-0.6)^n u[n]$.

7.6.2 Transfer Function

The transfer function plays an important role in the analysis of discrete-time linear systems. In this subsection, we will define the discrete-time transfer function and how to find the system impulse and step responses. We will also learn how to use the transfer function to determine system performance characteristics such as stability and frequency response.

The relationships between the system input $x[n]$, output $y[n]$, and impulse response and their z-transforms are given in eq. (7.35), namely

$$y[n]=x[n]*h[n]\leftrightarrow Y(z)=X(z)H(z) \tag{7.51}$$

From this, we obtain the transfer function $H(z)$ as

$$H(z)=\frac{Y(z)}{X(z)} \tag{7.52}$$

This implies that the transfer function is the ratio of the z-transforms of the output and input. We obtain the transfer function by either transforming the difference equation or transforming the system representation.

When the system input is a unit impulse $\delta[n]$ ($\delta[n]\leftrightarrow 1$), the output of the system is equal to the transfer function $H(z)$. Thus, the discrete-time system impulse response is given by

$$h[n]=\mathcal{Z}^{-1}\{H(z)\} \tag{7.53}$$

When the input $x[n]$ is a unit step $u[n]$ so that $X(z) = z/(z-1)$, the corresponding output y[n] is called the *step response*.

When systems are interconnected, the rules that apply to continuous-time systems also apply to discrete-time systems. We handle block diagrams such as series, parallel, and feedback interconnection in the same manner as for their corresponding

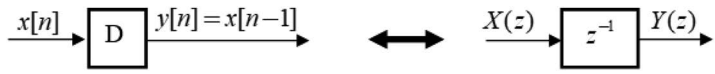


FIGURE 7.5 Unit delay.

continuous-time systems in Section 2.7. For a unit delay, shown in Figure 7.5, the input-output relationship is

$$y[n] = x[n-1] \tag{7.54a}$$

or

$$Y(z) = z^{-1}X(z) \tag{7.54b}$$

For the series connection shown in Figure 7.6(a), the overall transfer function is

$$H(z) = H_1(z)H_2(z) \tag{7.55}$$

For the parallel connection shown in Figure 7.6(b), the overall transfer function is

$$H(z) = H_1(z) + H_2(z) \tag{7.56}$$

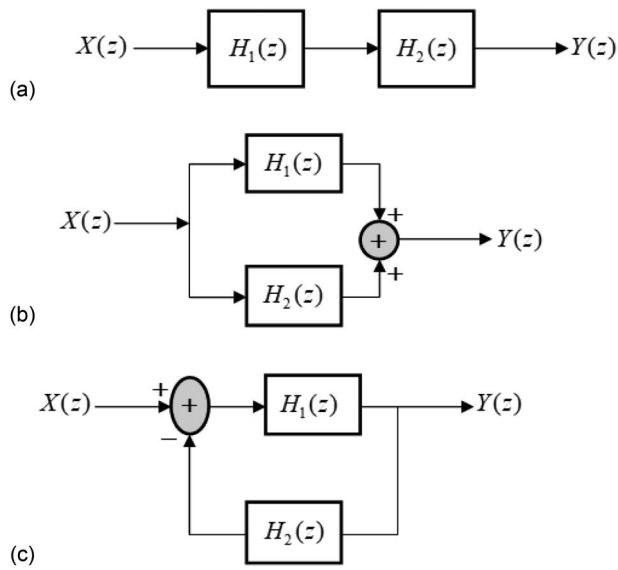


FIGURE 7.6 Transfer functions for (a) series connection, (b) parallel connection, and (c) feedback connection.

For the feedback interconnection shown in Figure 7.6(c), the overall transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)} \tag{7.57}$$

System stability may be addressed in terms of the characteristic roots, the impulse response, and the bounded-input/bounded-output (BIBO) criterion. It may also be deduced from the system transfer function. The system is stable if $h[n]$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{7.58}$$

All the terms in $h[n]$ are decaying exponentials. To achieve this, the magnitudes of all the system poles of $H(z)$ must be less than 1; i.e., the poles are inside a unit circle in the z-plane.

Example 7.17

Consider the α-filter, which is described by Figure 7.7. From the figure, it is evident that

$$y[n] + (1-\alpha)\, y[n-1] = \alpha x[n]$$

Find $H(z)$ and $h[n]$. You may take $\alpha = 0.2$.

SOLUTION

With $\alpha = 0.2$,

$$y[n] + 0.8y[n-1] = 0.2x[n]$$

We take the z-transform of the given difference equation to get

$$Y(z) + 0.8z^{-1}Y(z) = 0.2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.2}{1 + 0.8z^{-1}} = \frac{0.2z}{z + 0.8}$$

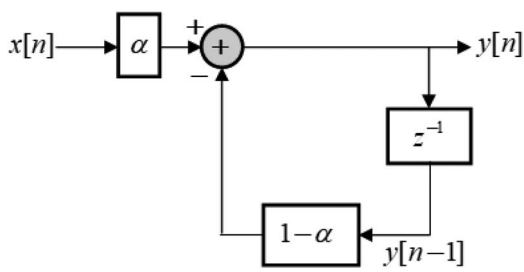


FIGURE 7.7 For Example 7.17.

Taking inverse z-transform gives

$$h[n] = 0.2(-0.8)^n u[n]$$

Practice Problem 7.17 Repeat the problem in Example 7.17 with

$$y[n] - 0.5y[n-1] = x[n]$$

Answer: $H(z) = \dfrac{z}{z-0.5}$, $h[n] = (0.5)^n u[n]$

Example 7.18

A discrete-time system is described by the difference equation

$$y[n] - 0.6y[n-1] - 0.05y[n-2] = x[n] - x[n-1]$$

Find the impulse response.

SOLUTION

Since there are no initial conditions, the z-transform of the difference equation is

$$Y(z) - 0.6\left\{z^{-1}Y(z)\right\} - 0.05\left\{z^{-2}\ Y(z)\right\} = X(z) - \left\{z^{-1}X(z)\right\}$$

The transfer function is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.6z^{-1} - 0.05z^{-2}} = \frac{z^2 - z}{z^2 - 0.6z - 0.05}$$

To find the impulse function, we apply a partial fraction on $H(z)$:

$$H_1(z) = \frac{\mathrm{H(z)}}{z} = \frac{z-1}{z^2 - 0.6z - 0.05} = \frac{z-1}{(z-0.5)(z-0.1)} = \frac{A}{z-0.5} + \frac{B}{z-0.1}$$

$$A = (z-0.5)H_1(z)\big|_{z=0.5} = \frac{0.5-1}{(0.5-0.1)} = -1.25$$

$$B = (z-0.1)H_1(z)\big|_{z=0.1} = \frac{0.1-1}{(1-0.5)} = -1.8$$

Thus,

$$H(z) = -\frac{1.25z}{z-0.5} - \frac{1.8z}{z-0.1}$$

Taking the inverse z-transform of this yields

$$h[n] = -1.25(0.5)^n u[n] - 1.8(0.1)^n u[n]$$

Practice Problem 7.18 Find the impulse response of the discrete-time system described by

$$y[n] - \frac{1}{12}y[n-1] - \frac{1}{12}y[n-2] = x[n-1] + x[n-2]$$

Answer: $-12\delta[n-1] + \frac{48}{7}\left(\frac{1}{3}\right)^n u[n] - \frac{36}{7}\left(\frac{1}{4}\right)^n u[n]$.

7.7 COMPUTING WITH MATLAB®

In this section, will discuss how MATLAB can be used to do the following:

- Find the z-transform of a discrete-time signal using the MATLAB command **ztrans**.
- Determine the inverse z-transform using the MATLAB command **iztrans**.
- Determine the poles of a transfer function using the MATLAB command **roots** and evaluate the stability of the system.
- Use the MATLAB commands **dstep** and **filter** to find the step response of a discrete-time linear system.

Example 7.19

Using MATLAB, find the z-transform of

$$x[n] = nu[n]$$

SOLUTION

The MATLAB script for doing this is:

```
syms  X x n z
x =  n*heaviside(n);
X = ztrans(x)
```

The program returns

```
X =
z/(z^2 - 2*z + 1)
```

which agrees with what we have in Table 7.2.

Practice Problem 7.19 Use MATLAB to find the z-transform of $y[n] = n^2u[n]$

Answer: $Y(z) = \frac{z^2 + z}{z^3 - 3z^2 + 3z - 1}$.

Example 7.20

Given the function

$$X(z) = \frac{6z + 4}{z^2 - 3.5z + 3}$$

Find the inverse z-transform $x[n]$.

SOLUTION

We can do this in two ways.

Method 1: Using the **iztrans** command, we have MATLAB script shown.

```
syms  X x n z
X = (6*z +4)/(z^2 -3.5*z + 3);
x = iztrans(X)
```

MATLAB returns

```
x =
16*2^n - (52*(3/2)^n)/3 + (4*kroneckerDelta(n, 0))/3
```

Method 2: We can also use the **residue** or **residuez** command, which carries out partial-fraction expansion. We illustrate it in the following script.

```
num = [ 6  4]
den = [1  -3.5  3]
[r, p, k] = residuez(num, den)
```

MATLAB responds as follows (r = residues, p = poles, k = direct terms).

```
r =
    32
   -26
p =
    2.0000
    1.5000
k =
     []
```

This implies that

$$X(z) = \frac{32}{1 - 2z^{-1}} - \frac{26}{1 - 1.5z^{-1}} = \frac{32z}{z - 2} - \frac{26z}{z - 1.5}$$

From this, we obtain the inverse z-transform as

$$x[n] = 32(2)^n u[n] - 25(1.5)^n u[n]$$

Practice Problem 7.20 Use MATLAB to find the inverse z-transform of

$$X(z) = \frac{z-2}{z^2+3z+2}$$

Answer: x[n] = 3*(–1)^n – 2*(–2)^n – kroneckerDelta(n, 0).

Example 7.21

The transfer function of a system is given by

$$H(z) = \frac{2z-3}{z^3-2z^3-z-0.5}$$

Determine whether the system is stable.

SOLUTION

We only need to find the roots of the denominator to determine the system stability.

```
den = [1  -2 -1  -0.5];
r = roots(den)
```

MATLAB returns

```
r =
   2.4837
  -0.2418 + 0.3779i
  -0.2418 - 0.3779i
```

The poles are located on the complex plane as shown in Figure 7.8. We notice that the magnitude of the first pole (2.4847) is greater than 1 and therefore lies outside the unit circle. (The other poles lie within the unit circle.) Hence, the system is unstable.

Practice Problem 7.21 Determine the stability of a system whose transfer function is

$$H(z) = \frac{2z-1}{z^3+0.5z^2+0.5z+0.5}$$

Answer: Stable; all poles lie within the unit circle.

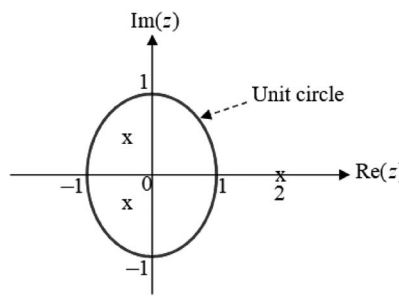


FIGURE 7.8 Pole locations for Example 7.21.

Example 7.22

Obtain the unit-step response of a discrete-time system whose transform is

$$H(z) = \frac{3+z}{z^2 - 5z + 2}$$

SOLUTION

The MATLAB script is as follows. We use the MATLAB command **filter** in this example, while the Practice Problem following this example uses the **dstep** command. Given the transfer function and the input and initial conditions, the filter returns the system output. The step response is shown in Figure 7.9.

```
num = [1  3];
den = [ 1 -5  2];
n = 0:1:30;
x = [1*ones(size(n))]; % unit step input
y = filter(num, den, x);
% d = length(y);
% n = 0:1:d-1
plot(n,y);
xlabel('Sample number n');
ylabel('step response y[n]')
```

Practice Problem 7.22 Change one line of the MATLAB script for Example 7.22 to y = dstep(num, den). You may delete two lines before that statement and add two lines commented after that statement. Plot your result.

Answer: See Figure 7.10.

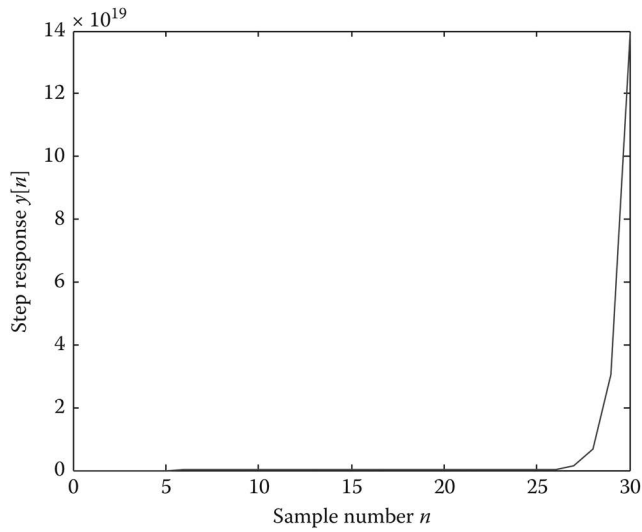


FIGURE 7.9 Unit-step response for Example 7.22.

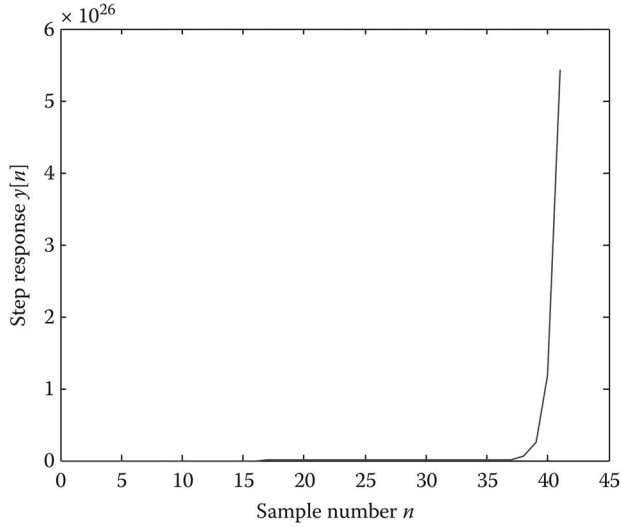


FIGURE 7.10 For Practice Problem 7.22.

SUMMARY

1. The z-transform is used for analyzing discrete-time signals, just as its counterpart, the Laplace transform, is used for analyzing continuous-time signals.
2. The (unilateral) z-transform of a signal $x[n]$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

3. The region of convergence of the z-transform consists of the values of z where $X(z)$ converges.
4. The properties of the z-transform are presented in Table 7.1. Table 7.2 provides the z-transform pairs of common signals.
5. To find the inverse z-transform of $X(z)$, we expand $X(z)$ in partial fractions and identify the inverse of each term using Table 7.2.
6. Among the applications of the z-transform are the solution of difference equation and the system transfer function.
7. The z-transform can be used to solve the difference equation of a system with initial conditions.
8. The transfer function $H(z)$ of a discrete-time system is

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{Y(z)}{X(z)}$$

where X(z) and Y(z) are respectively the z-transform of the input x[n] and output y[n].

The inverse z-transform of H(z) is the impulse response h[n].

REVIEW QUESTIONS

7.1 The z-transform may be regarded as the Laplace transform in disguise.
(a) True, (b) False

7.2 The z-transform of the sequence shown in Figure 7.11 is:
(a) $2z+z^2$, (b) $2z^{-1}+z^{-1}$, (c) $2z^{-1}+z^{-2}$, (d) $z^{-1}+2z^{-2}$

7.3 The z-transform of $\delta[n-1]$ is:
(a) 1, (b) z, (c) z^{-1}, (d) e^{-z}, (b) e^{z}

7.4 The z-transform for $u[n+1]$ is:
(a) $\dfrac{1}{z(z-1)}$, (b) $\dfrac{1}{(z-1)}$, (c) $\dfrac{z}{(z-1)}$, (d) $\dfrac{z^2}{(z-1)}$

7.5 Let $X(z)=\dfrac{z+1}{z^2+3z+2}$
The initial value $x[0]$ is:
(a) 0, (b) 1/2, (c) 1, (d) ∞

7.6 Refer to the function $X(z)$ in the previous Review Question. The final value $x[\infty]$ is:
(a) 0, (b) 1/2, (c) 1, (d) ∞

7.7 Given the difference equation

$$y[n]+y[n-1]+y[n-2]=u[n],\quad y[-1]=2,\quad y[-2]=1,$$

$y[0]$ is:
(a) –2, (b) –1, (c) 0, (d) 1, (e) 2

7.8 Which MATLAB command is used for finding the inverse z-transform?
(a) ztrans, (b) invztrans, (c) iztrans, (d) filter, (e) dstep

7.9 Which of these MATLAB commands is used for determining the stability of a system?
(a) freqz, (b) residue, (c) residuez, (d) roots

7.10 The pole plot of a system is shown in Figure 7.12. The system is:
(a) stable, (b) unstable, (c) marginally state.

Answers: 7.1 a, 7.2 c, 7.3 c, 7.4 d, 7.5 a, 7.6 a, 7.7 a, 7.8 c, 7.9 d, 7.10 a.

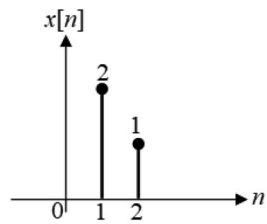


FIGURE 7.11 For Review Question 7.2.

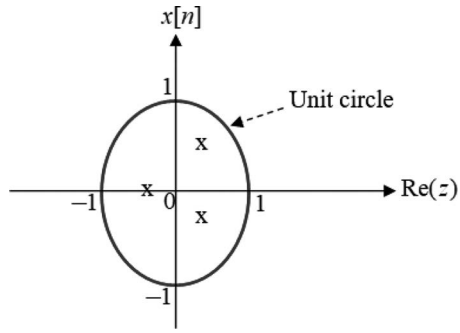


FIGURE 7.12 Pole locations for Review Question 7.10.

PROBLEMS

Sections 7.2 and 7.3: Definition and Region of Convergence

7.1 Find the z-transform of the sequences shown in Figure 7.13.
7.2 Determine the z-transform of the signal depicted in Figure 7.14.
7.3 Obtain the z-transform of the sequence in Figure 7.15.

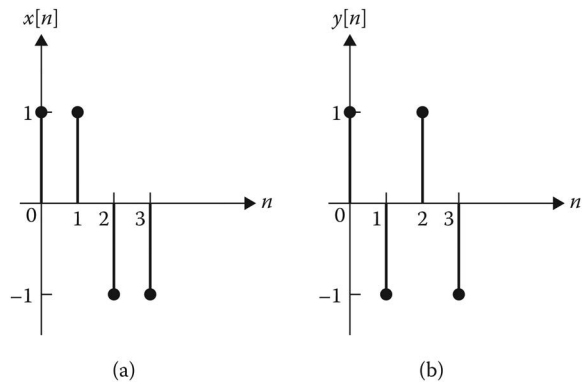


FIGURE 7.13 For Problem 7.1.

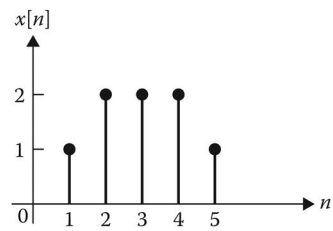


FIGURE 7.14 For Problem 7.2.

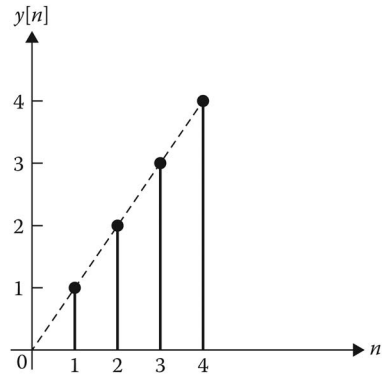


FIGURE 7.15 For Problem 7.3.

7.4 Determine the condition for the following summations to converge:

(a) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^2 z^n$

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n z^{-n}$

7.5 Determine the z-transform and its ROC for

$$x[n]=2\left(\frac{2}{3}\right)^n u[n]-\left(\frac{2}{5}\right)^n u[n]$$

7.6 Find the z-transform of

$$x[n]=2\delta[n]-4u[n]+\left(\frac{1}{3}\right)^n u[n]$$

Section 7.4: Properties of the z-Transform

7.7 Find the z-transform of the following signals:
(a) $u[n-m]$
(b) $na^n u[n]$
(c) $a^n \cos \pi n u[n]$

7.8 Determine the z-transform of each of the following signals:
(a) $u[n]-u[n-1]$
(b) $a^{n-1}\,u[n-1]$
(c) $na^n u[n-1]$

7.9 Find the z-transform of each of the following sequences:

(a) $x[n]=\left(\frac{1}{2}\right)^2+\left(-\frac{1}{3}\right)^n, \quad n=0,1,2,\ldots$

(b) $x[n]=\begin{cases}\left(\frac{1}{4}\right)^n, & n=0,1,2,\ldots\\ 0, & n=\text{negative}\end{cases}$

(c) $x[n]=\begin{cases}(0.5)^n, & 0\le n\le 5\\ 0, & \text{otherwise}\end{cases}$

7.10 Find the z-transform of the following sequences:

(a) $x[n] = na^n\cos(\Omega_0 n)u[n]$

(b) $x[n] = n^2u[n]$

(c) $x[n] = e^n\sin(n)u[n]$

7.11 Obtain the z-transform of the following signals:

(a) $x[n]=3\left(\frac{2}{5}\right)^n+4\left(-\frac{1}{3}\right)^n$

(b) $y[n] = 1 - e^{-an}$

7.12 Prove the following z-transform pairs.

(a) $(n+1)u[n] \Leftrightarrow \frac{z^2}{(z-1)^2}$

(b) $n^2a^nu[n] \Leftrightarrow \frac{az(z+a)}{(z-a)^3}$

(c) $a^2(\sin\Omega n)u[n] \Leftrightarrow \frac{za\sin\Omega}{z^2-2za\cos\Omega+a^2}$

7.13 Show that $\mathcal{Z}\{nx[n]\} = -z\frac{dX(z)}{dz}$.

7.14 Assuming that sequence $x[n]$ is real-valued, show that $X(z) = X^*(z^*)$.

7.15 A discrete-time signal $x[n] = A\cos(\Omega n)$ has the z-transform as

$$X(z)=\frac{4z(z-0.8)}{z^2-1.6z+1}$$

Compute the values of A and Ω.

7.16 Given that $x[n] = 3^nu[n]$, find the z-transform of

$$y[n]=nx[n]+x[n-1]u[n-1]+x[n+1]u[n+1]$$

7.17 Determine the initial values of the signals whose z-transforms are given in Problem 7.29.

7.18 Find the initial and final values of $x[n]$ for each of the following cases:

(a) $X(z) = \dfrac{2\left(z-\dfrac{1}{6}\right)}{\left(z-\dfrac{1}{4}\right)\left(z-\dfrac{1}{5}\right)}$

(b) $X(z) = \dfrac{z^2+1}{z^3+2z+2}$

7.19 The z-transform of $x[n]$ is given by

$$X(z) = \frac{2z+1}{(z-1)(z+1)}$$

(a) Find the final value $x[\infty]$.
(b) Determine the inverse z-transform of $X(z)$ and check your result in part (a).

7.20 Apply the initial value property to determine $x[0]$ for each of the following transforms:

(a) $X(z) = \dfrac{z^2}{z^2+z-1}$

(b) $X(z) = \dfrac{z^2+3z+4}{z^3+2z^2+z+1}$

7.21 Calculate the final value of $x[\infty]$ for

(a) $X(z) = \dfrac{z^2}{z^2+z-1}$

(b) $X(z) = \dfrac{z^2+3z+4}{z^3+2z^2+z+1}$

7.22 The z-transform of a discrete-time signal $x[n]$ is

$$X(z) = \frac{z-2}{z(z-1)}$$

Calculate $x[0]$, $x[1]$, and $x[10^5]$.

7.23 The z-transform of a discrete-time signal is

$$X(z) = \frac{2z}{z^2+3z+1}$$

Find the z-transform of the following signals:
(a) $y[n] = x[n-1]u[n-1]$
(b) $y[n] = \sin(\pi n/4)x[n]$
(c) $y[n] = n^n x[n]$
(d) $y[n] = 2x[n]*x[n]$

7.24 Determine the z-transform of the following signals:

(a) $x[n] = 2\sin(n\pi/4)u[n]$

(b) $y[n] = (0.2)^{(n-5)}u[n-5]$

(c) $h[n] = e^{-0.1n}\cos(n\pi/4)u[n]$

7.25 Let the z-transform of $x[n]$ be

$$X(z) = \frac{z^2}{z-3}$$

Find the z-transform of the following functions:

(a) $y[n] = x[n-2]$

(b) $z[n] = x[n+3]$

(c) $h[n] = nx[n]$

(d) $g[n] = n^2x[n]$

(e) $p[n] = 2^n x[n]$

7.26 Using the convolution property, find the z-transform of $x[n]$, where

$$x[n] = \sin\left(\frac{\pi n}{4}\right)u[n] * u[n]$$

7.27 Let $x[n] = [1, 1, 1, 1]$ and $y[n] = [1, -2, 1]$. Find the convolution of the two signals.

Section 7.5: Inverse z-Transform

7.28 Find the inverse z-transform of:

(a) $X_1(z) = \dfrac{1}{z^2 + \frac{5}{6}z + \frac{1}{6}}$

(b) $X_2 = \dfrac{z}{(z-1)(z-0.5)}$

(c) $X_3(z) = \dfrac{z^2 - 8}{(z-1)^2(z-2)}$

7.29 Find the inverse z-transform of the following functions:

(a) $X(z) = \dfrac{2z}{z^2 - z + 1}$

(b) $Y(z) = \dfrac{z(z+2)}{(z+1)(z-1)}$

(c) $H(z) = \dfrac{4z}{(z^2 + z + 1)(z + 1/2)}$

7.30 Obtain the inverse z-transform of

$$X(z) = \frac{z^2 + 2z - 10}{(z-1)(z+2)(z+3)}$$

7.31 Find the inverse z-transform of the following signals:

(a) $X(z) = \dfrac{z^2}{(z-3)(z-0.5)}$

(b) $Y(z) = \dfrac{4z^2+1}{z^2-5z+6}$

7.32 Determine the inverse z-transform of:

$$Y(z) = \frac{z(z+2)}{(z+1)(z-2)(z-3)}$$

7.33 Invert each of the following z-transforms:

(a) $X_1(z) = \dfrac{1-z^{-1}}{1-z^{-1}-0.75z^{-2}}$

(b) $X_2(z) = \dfrac{1+z^{-1}}{1-0.8z^{-1}+0.64z^{-2}}$

7.34 Use long division to find the inverse of

$$X(z) = \frac{z-2}{1-z+z^2}$$

Section 7.6: Applications

7.35 A discrete-time system is described by

$$y[n+2]+y[n+1]+0.36y[n]=0, \quad y[0]=2, \quad y[1]=1.$$

Find the response $y[n]$.

7.36 Obtain X(z) if

$$x[n]+0.5x[n-1]-0.25x[n-2]=0, \quad x[-1]=1=x[-2].$$

7.37 Using the z-transform, solve the following difference equation

$$Y[n+1]-2y[n]=(1.5)^n, \quad y[0]=1.$$

7.38 Find the transfer function of the discrete-time system represented by

$$y[n+2]+0.5\ y[n+1]+y[n]=x[n+1]+2x[n]$$

7.39 Obtain the complete response of the system represented by

$$y[n]+\frac{1}{3}y[n-1]+\frac{1}{6}y[n-2]=u[n], \quad y[-2]=0, \quad y[-1]=1$$

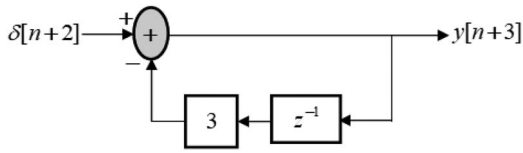


FIGURE 7.16 For Problem 7.43.

7.40 A digital filter has the impulse response

$$h[n]=\left(\frac{1}{2}\right)^n u[n-1]$$

Find the step response of the filter.

7.41 A discrete-time linear system has impulse response

$$h[n]=2+3(0.6)^n-5(-0.8)^n$$

Determine the transfer function.

7.42 The transfer function of a discrete-time system is

$$H(z)=\frac{1+2z^{-1}}{1-z^{-1}+z^{-2}}$$

Find the system response $y[n]$ when the input is a unit step function $u[n]$.

7.43 Find the z-transform of the output of the system shown in Figure 7.16.

7.44 Determine the transfer function of the feedback system represented in Figure 7.17.

7.45 Determine the overall transfer function of the system shown in Figure 7.18. Let

$$h_1[n]=\delta[n]+\delta[n-1],\quad h_2[n]=nu[n],\quad h_3[n]=\left(\frac{1}{3}\right)^n u[n].$$

7.46 A discrete system has input $x[n]$ and impulse response $h[n]$ as shown in Figure 7.19. Find the output $y[n]$ using z-transform.

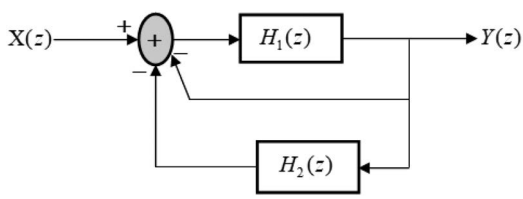


FIGURE 7.17 For Problem 7.44.

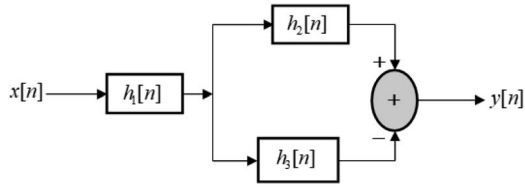
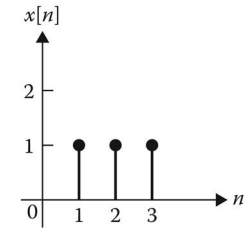
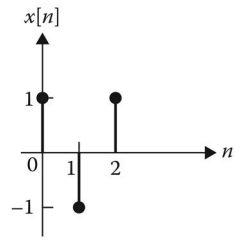


FIGURE 7.18 For Problem 7.45.



(a)



(b)

FIGURE 7.19 For Problem 7.46.

7.47 A discrete-time linear system is shown in Figure 7.20. (a) Find the transfer function $H(z)$ of the system. (b) Find the difference equation that describes the system.

7.48 Find the response of a system with a transfer function

$$H(z) = \frac{z - 0.6}{(z + 0.2)(z - 0.8)}$$

and an input $x[n]$ given by:
(a) $x[n] = u[n]$, (b) $x[n] = 2^n u[n]$.

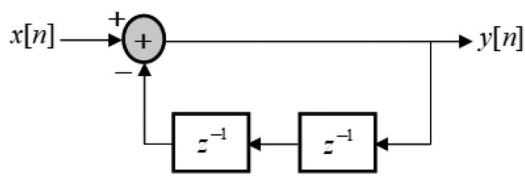


FIGURE 7.20 For Problem 7.47.

7.49 A discrete-time system has input $x[n] = \delta[n] + \delta[n-1]$ and output $y[n] = \delta[n] + \delta[n-1] + \delta[n-3]$. Find $H(z)$ and $h[n]$.

7.50 Obtain the impulse and step responses of the discrete-time system with transfer function

$$H(z) = \frac{0.8z}{(z-0.6)(z-2)}$$

Section 7.7: Computing with MATLAB

7.51 Use MATLAB to find the z-transform of the following signals:
(a) $x[n] = (0.8)^n u[n]$
(b) $x[n] = n(0.6)^n u[n]$

7.52 Use MATLAB to find the inverse z-transform of

$$X(z) = \frac{z}{z-0.6}$$

7.53 A linear discrete-time system is represented by the transfer function

$$H(z) = \frac{z+1}{z^3+2z^2+z+3}$$

Use MATLAB to plot the step response of the system.

7.54 Determine the poles and zeros of the transfer function:

$$H(z) = \frac{z^2+6z+z}{z^4+3z^{-3}+4z+10}$$

7.55 The z-transform of a system is

$$X(z) = \frac{z^3-0.6z^2+0.2z+1}{z^4+0.5z^3-z^2+0.5z+1}$$

Plot the locations of the poles and zeros of $X(z)$.

7.56 Determine the stability of the systems represented by the following transfer function:

$$H(z) = \frac{z^3+3z^2+z-1}{z^4+1.25z^3+0.5z^2-0.375z-0.2}$$

7.57 Check the stability of a system described by the following transfer function:

$$H(z) = \frac{z^{-3}-2z^2+6z+1}{z^5-2z^4+5z^2-z+4}$$

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Appendix A: Mathematical Formulas

This appendix—by no means exhaustive—serves as a handy reference. It does contain all the formulas needed to solve the problems in this book.

A.1 QUADRATIC FORMULAS

The roots of the quadratic equation $ax^2 + bx + c = 0$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A.2 TRIGONOMETRIC IDENTITIES

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\sec \mathrm{x} = \frac{1}{\cos\ \mathrm{x}}, \quad \csc x = \frac{1}{\sin x}$$

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

$$\sin(x \pm 90°) = \pm \cos x$$

$$\cos(x \pm 90°) = \mp \sin x$$

$$\sin(x \pm 180°) = -\sin x$$

$$\cos(x \pm 180°) = -\cos x$$

$$\cos^2 x + \sin^2 x = 1$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{(law of sines)}$$

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{(law of cosines)}$$

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b} \quad \text{(law of tangents)}$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$2\sin x \sin y = \cos(x-y) - \cos(x+y)$$

$$2\sin x \cos y = \sin(x+y) - \sin(x-y)$$

$$2\cos x \cos y = \cos(x+y) - \cos(x-y)$$

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$a\cos x + b\sin x = K\cos(x+\theta), \quad \text{where } K = \sqrt{a^2+b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

$$e^{\pm jx} = \cos x \pm j\sin x \text{ (Euler's formula)}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$1rad = 57.296°$$

A.3 HYBERBOLIC FUNCTIONS

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}$$

$$\text{csc h}x = \frac{1}{\sinh x}$$

$$\text{sec h}x = \frac{1}{\cosh x}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

A.4 DERIVATIVES

If $U = U(x)$, $V = V(x)$, and a = constant,

$$\frac{d}{dx}(aU) = a\frac{dU}{dx}$$

$$\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$$

$$\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

$$\frac{d}{dx}\left(aU^n\right) = naU^{n-1}$$

$$\frac{d}{dx}\left(a^U\right) = a^U \ln a \frac{dU}{dx}$$

$$\frac{d}{dx}\left(e^U\right) = e^U \frac{dU}{dx}$$

$$\frac{d}{dx}(\sin U) = \cos U \frac{dU}{dx}$$

$$\frac{d}{dx}(\cos U) = -\sin U \frac{dU}{dx}$$

$$\frac{d}{dx}\tan U = \frac{1}{\cos^2 U}\frac{dU}{dx}$$

A.5 INDEFINITE INTEGRALS

If $U = U(x)$, $V = V(x)$, and a = constant,

$$\int a\,dx = ax + C$$

$$\int U\,dV = UV - \int V\,dU \text{ (integration by parts)}$$

$$\int U^n\,dU = \frac{U^{n+1}}{n+1} + C, \quad n \neq 1$$

$$\int \frac{dU}{U} = 1\mathrm{n}U + C$$

$$\int a^U \, dU = \frac{a^U}{1\mathrm{n}a} + C, \quad a > 0, \quad a \neq 1$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

$$\int xe^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int x^2 e^{ax} \, dx = \frac{e^{ax}}{a^3}\left(a^2x^2 - 2ax + 2\right) + C$$

$$\int 1\mathrm{n}x \, dx = x1\mathrm{n}x - x + C$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax + C$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$$

$$\int x \sin ax \, dx = \frac{1}{a^2}(\sin ax - ax\cos ax) + C$$

$$\int x \cos ax \, dx = \frac{1}{a^2}(\cos ax + ax\sin ax) + C$$

$$\int x^2 \sin ax \, dx = \frac{1}{a^3}\left(2ax\sin ax + 2\cos ax - a^2x^2\cos ax\right) + C$$

$$\int x^2 \cos ax \, dx = \frac{1}{a^3}\left(2ax\cos ax - 2\sin ax + a^2x^2\sin ax\right) + C$$

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$$

$$\int \sin ax \sin bx\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$\int \sin ax \cos bx\, dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$\int \cos ax \cos bx\, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C, \quad a^2 \neq b^2$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{x^2 dx}{a^2 + x^2} = x - a \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{dx}{\left(a^2 + x^2\right)^2} = \frac{1}{2a^2}\left(\frac{x}{x^2 + a^2} + \frac{1}{a} \tan^{-1} \frac{x}{a}\right) + C$$

A.6 DEFINITE INTEGRALS

If m and n are integers,

$$\int_0^{2\pi} \sin ax\ dx = 0$$

$$\int_0^{2\pi} \cos ax\ dx = 0$$

$$\int_0^{\pi} \sin^2 ax\ dx = \int_0^{\pi} \cos^2 ax dx = \frac{\pi}{2}$$

$$\int_0^{\pi} \sin mx \sin nx dx = \int_0^{\pi} \cos mx \cos nx dx = 0, \quad m \neq n$$

$$\int_0^{\pi} \sin mx \cos nx dx = \begin{cases} 0, & m+n = \text{even} \\ \dfrac{2m}{m^2 - n^2}, & m+n = \text{odd} \end{cases}$$

$$\int_0^{2\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m \neq n \end{cases}$$

$$\int_0^{\infty} \frac{\sin ax}{x} dx = \begin{cases} \dfrac{\pi}{2}, & a > 0 \\ 0, & a = 0 \\ -\dfrac{\pi}{2}, & a < 0 \end{cases}$$

$$\int_0^{\infty} \frac{\sin^2 x}{x} dx = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\cos bx}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-ab}, \quad a > 0, \quad b > 0$$

$$\int_0^{\infty} \frac{x \sin bx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ab}, \quad a > 0, \quad b > 0$$

$$\int_0^{\infty} \sin cx dx = \int_0^{\infty} \sin c^2 x dx = \frac{1}{2}$$

$$\int_0^{\pi} \sin^2 nx dx = \int_0^{\pi} \sin^2 x dx = \int_0^{\pi} \cos^2 nx dx = \int_0^{\pi} \cos^2 x dx = \frac{\pi}{2}, \quad n = \text{an integer}$$

$$\int_0^{\pi} \sin mx \sin nx dx = \int_0^{\pi} \cos mx \cos nx dx = 0, \quad m \neq n, m, n \text{ integers}$$

$$\int_0^{\pi} \sin mx \cos nx dx = \begin{cases} \dfrac{2m}{m^2 - n^2}, & m+n = \text{odd} \\ 0, & m+n = \text{even} \end{cases}$$

$$\int_{-\infty}^{\infty} e^{\pm j2\pi tx} dx = \delta(t)$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}, \quad a > 0$$

$$\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad a > 0$$

A.7 L'HOPITAL'S RULE

If $f(0) = 0 = h(0)$, then

$$\lim_{x \to 0} \frac{f(x)}{h(x)} = \lim_{x \to 0} \frac{f'(x)}{h'(x)}$$

where the prime indicates differentiation.

A.8 TAYLOR AND MACLAURIN SERIES

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

where the prime indicates differentiation.

A.9 POWER SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$
$$\approx 1 + nx, \quad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1$$

$$Q(x) = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}\left(1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} - \frac{1\cdot 3\cdot 5}{x^6} + \cdots\right)$$

$$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \cdots$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right), \quad x \gg 1$$

$$I_0(x) \approx \begin{cases} e^{x^2/4}, & x^2 \ll 1 \\ \dfrac{e}{\sqrt{2\pi x}}, & x \gg 1 \end{cases}$$

A.10 SUMS

$$\sum_{k=1}^{N} k = \frac{1}{2} N(N+1)$$

$$\sum_{k=1}^{N} k^2 = \frac{1}{6} N(N+1)(2N+1)$$

$$\sum_{k=1}^{N} k^3 = \frac{1}{4} N^2 (N+1)^2$$

$$\sum_{k=0}^{N} a^k = \frac{a^{N+1} - 1}{a - 1} \, a \neq 1$$

$$\sum_{k=M}^{N} a^k = \frac{a^{N+1} - a^M}{a - 1} \, a \neq 1$$

$$\sum_{k=0}^{N} \binom{N}{k} a^{N-k} b^k = (a+b)^N, \quad \text{where} \binom{N}{k} = \frac{N!}{(N-k)!\,k!}$$

A.11 LOGARITHMIC IDENTITIES

$$\log xy = \log x + \log y$$

$$\log \frac{x}{y} = \log x - \log y$$

$$\log x^n = n \log x$$

$$\log_{10} x = \log x \text{ (common logarithm)}$$

$$\log_e x = \ln x \text{ (natural logarithm)}$$

A.12 EXPONENTIAL IDENTITIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

where $e = 2.7182$

$$e^x e^y = e^{x+y}$$

$$(e^x)^n = e^{nx}$$

$$\ln e^x = x$$

A.13 APPROXIMATIONS

If $|x| \ll 1$,

$$(1 \pm x)^n \simeq 1 \pm nx$$

$$e^x \simeq 1 + x$$

$$\ln(1 + x) \simeq x$$

$$\sin x = x \text{ or } \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\cos x \simeq 1$$

$$\tan x \simeq x$$

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Appendix B: Complex Numbers

The ability to handle complex numbers is important in signals and systems. Although calculators and computer software packages such as MATLAB® are now available to manipulate complex numbers, it is advisable that students be familiar with how to handle them by hand.

B.1 REPRESENTATION OF COMPLEX NUMBERS

A complex number z may be written in *rectangular form* as

$$z = x + jy \tag{B.1}$$

where $j = \sqrt{-1}$; x is the real part of z, while y is the imaginary part; that is

$$x = Re(z), \quad y = Im(z) \tag{B.2}$$

The complex number z is shown plotted in the complex plane in Figure B.1. Since $j = \sqrt{-1}$

$$\begin{aligned} \frac{1}{j} &= -j \\ j^2 &= -1 \\ j^3 &= j \cdot j^2 = -j \\ j^4 &= j^2 \cdot j^2 = 1 \\ j^5 &= j \cdot j^4 = j \\ &\vdots \end{aligned} \tag{B.3}$$

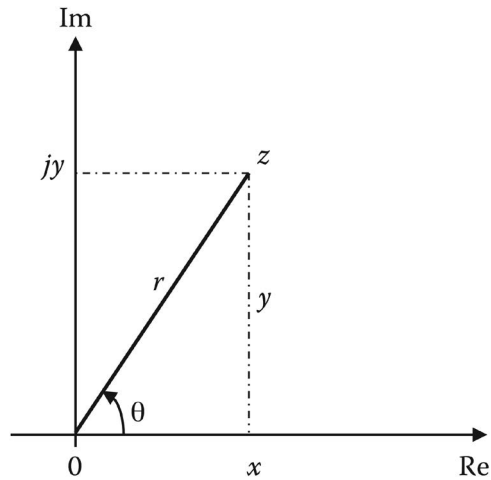


FIGURE B.1 Graphical representation of a complex number.

A second way of representing the complex number z is by specifying with magnitude r and angle θ, it makes the real axis, as shown in Figure B.1. This is known as the *polar form*. It is given by

$$z = |z| \angle \theta = r \angle \theta \tag{B.4}$$

where

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x} \tag{B.5a}$$

or

$$x = r \cos \theta, \quad y = r \sin \theta \tag{B.5b}$$

that is,

$$z = x + jy = r \angle \theta = r \cos \theta + jr \sin \theta \tag{B.6}$$

In converting from rectangular to polar form using eq. (B.5), we must exercise care in determining the correct value of θ. These are the four possibilities:

$$\begin{aligned}
z &= x + jy, \quad \theta = \tan^{-1} \frac{y}{x} && \text{(1st quadrant)} \\
z &= -x + jy, \quad \theta = 180^\circ - \tan^{-1} \frac{y}{x} && \text{(2nd quadrant)} \\
z &= -x - jy, \quad \theta = 180^\circ + \tan^{-1} \frac{y}{x} && \text{(3rd quadrant)} \\
z &= x - jy, \quad \theta = 360^\circ - \tan^{-1} \frac{y}{x} && \text{(4th quadrant)}
\end{aligned} \tag{B.7}$$

assuming that x and y are positive.

The third way of representing the complex number x is the *exponential form*:

$$z = re^{j\theta} \tag{B.8}$$

This is almost the same as the polar form, because we use the same magnitude r and the angle θ.

The three forms of representing a complex number are summarized as follows.

$$\begin{aligned}
&z = x + jy, \quad (x = r\cos\theta, y = r\sin\theta) && \text{Rectangular form} \\
&z = r\angle\theta, \quad \left(r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}\right) && \text{Polar form} \\
&z = re^{j\theta}, \quad \left(r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}\right) && \text{Exponential form}
\end{aligned} \tag{B.9}$$

B.2 MATHEMATICAL OPERATIONS

Two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ are equal if and only if their real parts are equal and their imaginary parts are equal, that is

$$x_1 = x_2, \quad y_1 = y_2 \tag{B.10}$$

The complex conjugate of the complex number $z = x + jy$ is

$$z^* = x - jy = r\angle -\theta = re^{-j\theta} \tag{B.11}$$

Thus, the complex conjugate of a complex number is found by replacing every j by $-j$.

Given two complex numbers $z_1 = x_1 + jy_1 = r_1\angle\theta_1$ and $z_2 = x_2 + jy_2 = r_2\angle\theta_2$, their sum is

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{B.12}$$

and their difference is

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{B.13}$$

While it is more convenient to perform addition and subtraction of complex numbers in rectangular form, the product and quotient of two complex numbers are best done in polar or exponential form. For their product,

$$z_1 z_2 = r_1 r_2 \angle\theta_1 + \theta_2 \tag{B.14}$$

Alternatively, using the rectangular form

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned} \tag{B.15}$$

For their quotient,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \theta_1 - \theta_2 \tag{B.16}$$

Alternatively, using the rectangular form,

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \tag{B.17}$$

we rationalize the denominator by multiplying both the numerator and denominator by z_2^*.

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^3} + j\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^3} \tag{B.18}$$

B.3 EULER'S FORMULA

Euler's formula is an important result in complex variables. We derive it from the series expansion of e^x, $\cos\theta$, and $\sin\theta$. We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{B.19}$$

Replacing x by $j\theta$ gives

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \tag{B.20}$$

Also,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{B.21a}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{B.21b}$$

so that

$$\cos\theta + j\sin\theta = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \cdots \tag{B.22}$$

Comparing eqs. (B.20) and (B.22), we conclude that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.23}$$

This is known as Euler's formula. The exponential form of representing a complex number as in eq. (B.8) is based on Euler's formula. From eq. (B.23), notice that

$$\cos\theta = \text{Re}(e^{j\theta}), \quad \sin\theta = \text{Im}(e^{j\theta}) \tag{B.24}$$

and that

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \tag{B.25}$$

Replacing θ by $-\theta$ in eq. (B.23) gives

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{B.26}$$

Adding eqs. (B.23) and (B.26) yields

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{B.27}$$

Subtracting eq. (B.26) from eq. (B.23) yields

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \tag{B.28}$$

The following identities are useful in dealing with complex numbers. If $z = x + jy = r\angle\theta$, then

$$zz^* = |z|^2 = x^2 + y^2 = r^2 \tag{B.29}$$

$$\sqrt{z} = \sqrt{x + jy} = \sqrt{r}e^{j\theta/2} = \sqrt{r}\angle\theta/2 \tag{B.30}$$

$$z^n = (x + jy)^n = r^n\angle n\theta = r^n(\cos n\theta + j\sin n\theta) \tag{B.31}$$

$$z^{1/n} = (x + jy)^{1/n} = r^{1/n} \angle \theta/n + 2\pi k/n, \quad k = 0, 1, 2, \ldots, n-1 \tag{B.32}$$

$$\ln(re^{j\theta}) = \ln r + \ln e^{j\theta} = \ln r + j\theta + j2\pi k \text{ (k = integer)} \tag{B.33}$$

$$\begin{aligned} e^{\pm j\pi} &= -1 \\ e^{\pm j2\pi} &= 1 \\ e^{j\pi/2} &= j \\ e^{-j\pi/2} &= -j \\ \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned} \tag{B.34}$$

$$\begin{aligned} \text{Re}\left(e^{(\alpha + j\omega)t}\right) &= \text{Re}\left(e^{\alpha t} e^{j\omega t}\right) = e^{\alpha t} \cos \omega t \\ \text{Im}\left(e^{(\alpha + j\omega)t}\right) &= \text{Im}\left(e^{\alpha t} e^{j\omega t}\right) = e^{\alpha t} \sin \omega t \end{aligned} \tag{B.35}$$

MATLAB handles complex numbers quite easily as real numbers.

Example B.1

Evaluate the complex numbers:

a. $z_1 = \dfrac{j(3 - j4)^*}{(-1 + j6)(2 + j)^2}$

b. $z_2 = \left[4\angle 30^\circ + 2 - j + 6e^{j\pi/4}\right]^{1/2}$

SOLUTION

a. This can be solved in two ways: working with z in rectangular form or polar form.

Method 1 (working in rectangular form):

Let $z_1 = \dfrac{z_3 z_4}{z_5 z_6}$

where

$z_3 = j$

$z_4 = (3 - j4)^* =$ the complex conjugate of $(3 - j4) = (3 + j4)$

$z_5 = -1 + j6$

$z_6 = (2 + j)^2 = 4 + j4 - 1 = 3 + j4$

Hence,

$$z_1 = \frac{j(3+j4)}{(-1+j6)(3+j4)} = \frac{j3-4}{-3-j4+j18-24}$$
$$= \frac{-4+j3}{-27+j14}$$

Multiplying and dividing by $-27 - j14$ (rationalization), we have

$$z_1 = \frac{(-4+j3)(-27-j14)}{(-27+j14)(-27-j14)} = \frac{150-j25}{27^2+14^2}$$
$$= 0.1622 - j0.027 = 0.1644\angle -9.46°$$

Method 2 (working in polar form)

$$z_3 = j = 1\angle 90°$$
$$z_4 = (3-j4)^* = (5\angle -53.13°)^* = 5\angle 53.13°$$
$$z_5 = (-1+j6) = \sqrt{37}\angle 99.46°$$
$$z_6 = (2+j)2 = (\sqrt{5}\angle 26.56°)^2 = 5\angle 53.13°$$

Hence,

$$z_1 = \frac{z_3 z_4}{z_5 z_6} = \frac{(1\angle 90°)(5\angle 53.13°)}{(\sqrt{37}\angle 99.46°)(5\angle 53.13°)} = \frac{1}{\sqrt{37}}\angle(90° - 99.46°)$$
$$= 0.1644\angle -9.46° = 0.1622 - j0.027$$

b. Let

$$z_7 = 4\angle 30° = 4\cos 30° + j4\sin 30° = 3.464 + j2$$
$$z_8 = 2 - j$$
$$z_9 = 6e^{j\pi/4} = 6\cos 45° + j6\sin 45° = 4.243 + j4.243$$

Then,

$$z_2 = \left[z_7 + z_8 + z_9\right]^{1/2}$$
$$= \left[3.464 + j2 + 2 - j + 4.243 + j4.243\right]^{1/2}$$
$$= (9.707 + j5.243)^{1/2} = (11.03\angle 28.374°)^{1/2}$$
$$= 3.32\angle 14.19°$$

Practice Problem B.1 Evaluate the following complex numbers:

a. $j^3\left[\dfrac{1+j}{2-j}\right]^2$

b. $6\angle 30° + j5 - 3 + e^{j45°}$

Answer: (a) $0.24 + j0.32$, (b) $2.03 + j8.707$.

Appendix C: MATLAB®

MATLAB® has become a powerful tool of technical professionals worldwide. The term MATLAB is an abbreviation for MATrix LABoratory implying, that MATLAB is a computational tool that employs matrices and vectors/arrays to carry out numerical analysis, signal processing, and scientific visualization tasks. Because MATLAB uses matrices as its fundamental building blocks, one can write mathematical expressions involving matrices just as easily as one would on paper. MATLAB is available for Macintosh, Unix, and Windows operating systems. A student version of MATLAB is available for PCs. A copy of MATLAB can be obtained from:

The Mathworks, Inc.
3 Apple Hill Drive
Natick, MA 01760-2098
Phone:(508) 647-7000
Website: http://www.mathworks.com

A brief introduction on MATLAB is presented in this appendix. What is presented is sufficient for solving the problems in this book. Other information on MATLAB required in this book is provided on chapter-to-chapter basis as needed. Additional information about MATLAB can be found in MATLAB books and from online help. The best way to learn MATLAB is to work with it after one has learned the basics.

C.1 MATLAB FUNDAMENTALS

The Command window is the primary area where you interact with MATLAB. A little later, we will learn how to use the text editor to create M-files, which allow executing sequences of commands. For now, we focus on how to work in the Command window. We will first learn how to use MATLAB as a calculator. We do so by using the algebraic operators in Table C.1.

To begin to use MATLAB, we use these operators. Type commands to the MATLAB prompt ">>" in the Command window (correct any mistakes by backspacing) and press the <Enter> key. For example,

```
» a=2; b=4; c=-6;
» dat = b^2 - 4*a*c
dat =
    64
» e=sqrt(dat)/10
e =
    0.8000
```

TABLE C.1
Basic Operations

Operation	MATLAB Formula
Addition	a + b
Division (right)	a/b (means $a \div b$)
Division (left)	a\b (means $b \div a$)
Multiplication	a*b
Power	a^b
Subtraction	a – b

The first command assigns the values 2, 4, and –6 to the variables *a, b*, and *c*, respectively. MATLAB does not respond because this line ends with a colon. The second command sets *dat* to $b^2 - 4ac$, and MATLAB returns the answer as 64. Finally, the third line sets *e* equal to the square root of *dat* and divides by 10. MATLAB prints the answer as 0.8. As function *sqrt* is used here, other mathematical functions listed in Table C.2 can be used. Table C.2 provides just a small sample of MATLAB functions. Others can be obtained from the online help. To get help, type

```
>> help
```

TABLE C.2
Typical Elementary Math Functions

Function	Remark
abs(x)	Absolute value or complex magnitude of *x*
acos, acosh(x)	Inverse cosine and inverse hyperbolic cosine of *x* in radians
acot, acoth (x)	Inverse cotangent and inverse hyperbolic cotangent of *x* in radians
angle(x)	Phase angle (in radian) of a complex number *x*
asin, asinh(x)	Inverse sine and inverse hyperbolic sine of *x* in radians
atan, atanh(x)	Inverse tangent and inverse hyperbolic tangent of *x* in radians
conj(x)	Complex conjugate of *x*
cos, cosh(x)	Cosine and hyperbolic cosine of *x* in radian
cot, coth(x)	Cotangent and hyperbolic cotangent of *x* in radian
exp(x)	Exponential of *x*
fix	Round toward zero
imag(x)	Imaginary part of a complex number *x*
log (x)	Natural logarithm of *x*
log2(x)	Logarithm of *x* to base 2
log10(x)	Common logarithms (base 10) of *x*
real (x)	Real part of a complex number *x*
sin, sinh(x)	Sine and hyperbolic sine of *x* in radian
sqrt (x)	Square root of *x*
tan, tanh	Tangent and hyperbolic tangent of *x* in radian

[a long list of topics comes up]

and for a specific topic, type the command name. For example, to get help on *log to base 2*, type

```
>> help log2
```

[a help message on the log function follows]
Note that MATLAB is case sensitive so that sin(a) is not the same as sin(A).
Try the following examples:

```
>> 3^(log10(25.6))
>> y=2* sin(pi/3)
>>exp(y+4-1)
```

In addition to operating on mathematical functions, MATLAB easily allows one to work with vectors and matrices. A vector (or array) is a special matrix with one row or one column. For example,

```
>> a = [ 1  -3  6  10  -8  11   14];
```

is a row vector. Defining a matrix is similar to defining a vector. For example, a 3×3 matrix can be entered as

```
>>  A = [ 1  2 3; 4  5  6; 7  8  9]
```

or as

```
>> A=[1  2  3
      4  5  6
      7  8  9]
```

In addition to the arithmetic operations that can be performed on a matrix, the operations in Table C.3 can be implemented.

TABLE C.3
Matrix Operations

Operation	Remark
A′	Finds the transpose of matrix *A*
det(A)	Evaluates the determinant of matrix *A*
inv(A)	Calculates the inverse of matrix *A*
eig(A)	Determines the eigenvalues of matrix *A*
diag(A)	Finds the diagonal elements of matrix *A*
expm(A)	Exponential of matrix *A*

Using the operations in Table C.3, we can manipulate matrices as follows.

```
» B = A'
      1     4     7
B =   2     5     8
      3     6     9

» C = A + B
      2     6    10
C =   6    10    14
     10    14    18

» D = A^3 - B*C
      372   432   492
D =   948   1131  1314
      1524  1830  2136

» e =[1 2; 3 4]
      1  2
e =
      3  4

» f=det(e)
f=-2

» g = inv(e)
      -2.0000   1.0000
g =
       1.5000  -0.5000

» H = eig(g)
      -2.6861
H =
       0.1861
```

Note that not all matrices can be inverted. A matrix can be inverted if and only if its determinant is nonzero. Special matrices, variables, and constants are listed in Table C.4. For example, type

```
>> eye(3)
```

$$\text{ans} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

to get a 3×3 identity matrix.

TABLE C.4
Special Matrices, Variables, and Constants

Matrix/Variable/Constant	Remark
eye	Identity matrix
ones	An array of ones
zeros	An array of zeros
i or *j*	Imaginary unit or sqrt(–1)
pi	3.142
NaN	Not a number
inf	Infinity
eps	A very small number, $2.2e^{-16}$
rand	Random element

C.2 USING MATLAB TO PLOT

To plot using MATLAB is easy. For a two-dimensional plot, use the plot command with two arguments as:

```
>> plot(xdata,ydata)
```

where *xdata* and *ydata* are vectors of the same length containing the data to be plotted.

For example, suppose we want to plot *y* = 10*sin(2*pi**x*) from 0 to 5*pi. We will proceed with the following commands:

```
>> x = 0:pi/100:5*pi;     % x is a vector, 0 <= x <= 5*pi,
                          increments of pi/100
>> y = 10*sin(2*pi*x);    % create a vector y
>> plot(x,y);             % create the plot
```

With this, MATLAB responds with the plot in Figure C.1.

MATLAB will let you graph multiple plots together and distinguish them with different colors. This is obtained with the command **plot**(xdata, ydata, 'color'), where the color is indicated by using a character string from the options listed in Table C.5.

For example,

```
>> plot(x1, y1, 'r', x2,y2, 'b', x3,y3, '--');
```

will graph data (x1,y1) in red, data (x2,y2) in blue, and data (x3,y3) in a dashed line, all on the same plot.

MATLAB also allows for logarithm scaling. Rather than the **plot** command, we use:

```
loglog     log(y) versus log(x)
semilogx   y versus log(x)
semilogy   log(y) versus x
```

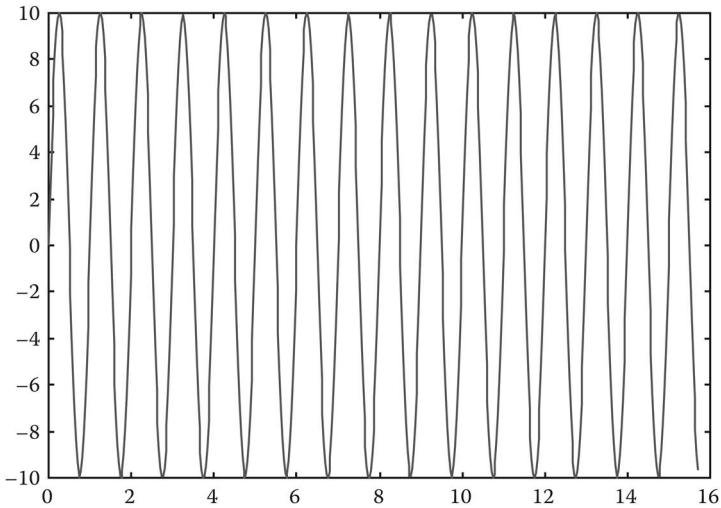


FIGURE C.1 MATLAB plot of y = 10*sin(2*pi*x).

Three-dimensional plots are drawn using the functions *mesh* and *meshdom* (mesh domain). For example, to draw the graph of $z = x$*exp(–x^2 – y^2) over the domain –1 < x, y < 1, we type the following commands:

```
>> xx = -1:.1:1;
»  yy = xx;
»  [x,y] = meshgrid(xx,yy);
»  z=x.*exp(-x.^2 -y.^2);
» mesh(z);
```

(The dot symbol used in x. and y. allows element-by-element multiplication.) The result is shown in Figure C.2.

TABLE C.5
Various Color and Line Types

y	yellow	.	point
m	magenta	o	circle
c	cyan	x	x-mark
r	red	+	plus
g	green	-	solid
b	blue	*	star
w	white	:	dotted
k	black	-.	dashdot
		--	dashed

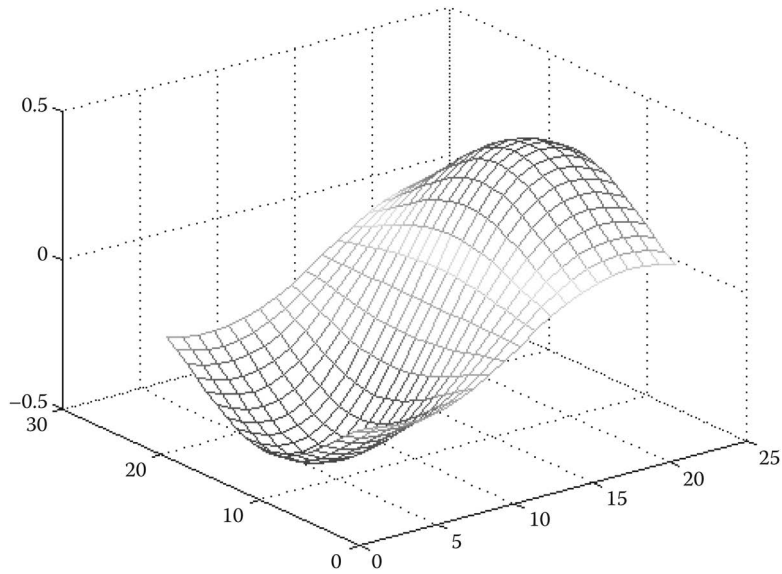


FIGURE C.2 A three-dimensional plot.

Other plotting commands in MATLAB are listed in Table C.6. The **help** command can be used to find out how each of these is used.

C.3 PROGRAMMING WITH MATLAB

So far MATLAB has been used as a calculator, but you can also use MATLAB to create your own program. The command-line editing in MATLAB can be inconvenient if one has several lines to execute. To avoid this problem, one creates a program, which is a sequence of statements to be executed. If you are in the Command window, click **File/New/M-files** to open a new file in the MATLAB Editor/Debugger or

TABLE C.6
Other Plotting Commands

Command	Comments
bar(x,y)	A bar graph
contour(z)	A contour plot
errorbar (x,y,l,u)	A plot with error bars
hist(x)	A histogram of the data
plot3(x,y,z)	A three-dimensional version of plot()
polar(r, angle)	A polar coordinate plot
stairs(x,y)	A stairstep plot
stem(n,x)	Plots the data sequence as stems
subplot(m,n,p)	Multiple (m-by-n) plots per window
surf(x,y,x,c)	A plot of 3D colored surface

simple text editor. Type the program and save the program in a file with an extension. m, say, filename.m; it is for this reason it is called an M-file. Once the program is saved as an M-file, exit the Debugger window. You are now back in the Command window. Type the file without the extension .m to get the results. For example, the plot that was made earlier can be improved by adding a title and labels and typed as an M-file called example1.m.

```
x = 0:pi/100:5*pi;              % x is a vector, 0 <= x <= 5*pi,
                                increments of pi/100
y = 10*sin(2*pi*x);             % create a vector y
plot(x,y);                      % create the plot
xlabel('x  (in radians)');      % label the x-axis
ylabel('10*sin(2*pi*x)');       % label the y-axis
title('A sine functions');      % title the plot
grid                            % add grid
```

Once it is saved as *example1.m* and we exit the text editor, type

```
>> example1
```

in the Command window and hit <Enter> to obtain the result shown in Figure C.3.

To allow flow control in a program, certain relational and logical operators are necessary. They are shown in Table C.7. Perhaps the most commonly used flow control

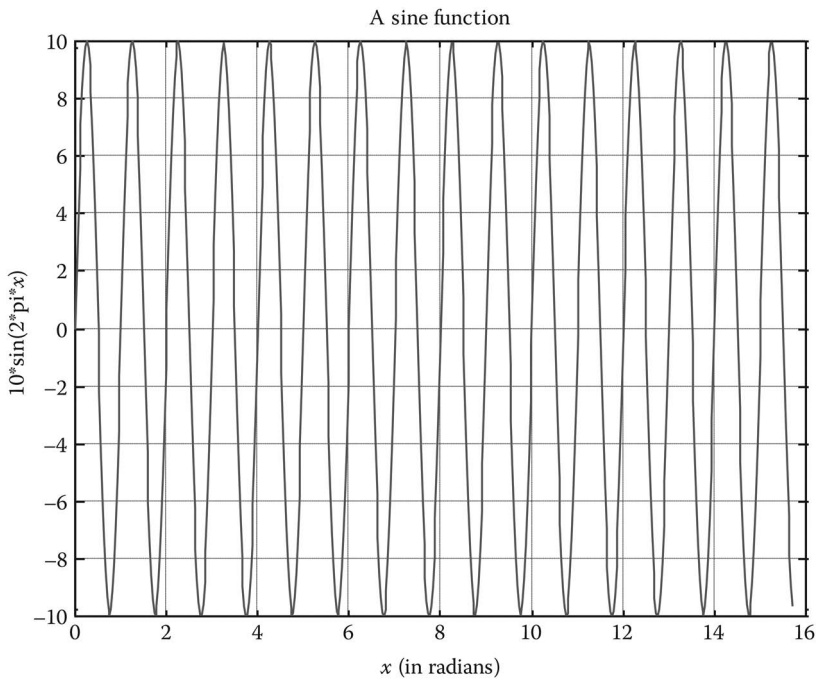


FIGURE C.3 MATLAB plot of y = 10*sin(2*pi*x) with title and labels.

TABLE C.7
Relational and Logical Operators

Operator	Remark
<	Less than
<=	Less than or equal
>	Greater than
>=	Greater than or equal
= =	Equal
~ =	Not equal
&	And
|	Or
~	Not

statements are *for* and *if*. The *for* statement is used to create a loop or a repetitive procedure and has the general form

```
for  x = array
    [commands]
end
```

The *if* statement is used when certain conditions need to be met before an expression is executed. It has the general form

```
if  expression
    [commands if expression is True]
else
   [commands if expression is False]
end
```

For example, suppose we have an array $y(x)$ and we want to determine the minimum value of y and its corresponding index x. This can be done by creating an M-file as shown:

```
%  example2.m
% This program finds the minimum y value and its corresponding
x index
x = [1 2 3  4 5 6  7  8 9 10]; %the nth term in y
y = [3 9 15 8 1 0 -2  4 12 5];
min1 = y(1);
for k=1:10
   min2=y(k);
   if(min2 < min1)
      min1 = min2;
      xo = x(k);
   else
      min1 = min1;
   end
```

```
end
diary
min1, xo
diary off
```

Note the use of *for* and *if* statements. When this program is saved as example2.m, we execute it in the Command window and obtain the minimum value of y as –2 and the corresponding value of x as 7, as expected.

```
» example2
min1 =
    -2
xo =
     7
```

If we are not interested in the corresponding index, we could do the same thing using the command

```
>> min(y)
```

The following tips are helpful in working effectively with MATLAB:

- Comment your M-file by adding lines beginning with a % character.
- To suppress output, end each command with a semicolon (;). You may remove the semicolon when debugging the file.
- Press up and down arrow keys to retrieve previously executed commands.
- If your expression does not fit on one line, use an ellipse (…) at the end of the line and continue on the next line. For example, MATLAB considers

$$\mathrm{y} = \sin\left(\mathrm{x} + \log10\left(2\mathrm{x}+3\right)\right) + \cos(\mathrm{x} + \cdots$$
$$\log10\left(2\mathrm{x}+3\right);$$

  as one line of expression.
- Keep in mind that variable and function names are case sensitive.

C.4 SOLVING EQATIONS

Consider the general system of n simultaneous equations as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \quad \cdots \quad \cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

or in matrix form

$$AX = B$$

where

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{bmatrix}, \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \quad B=\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

A is a square matrix and is known as the coefficient matrix, while X and B are vectors. X is the solution vector we are seeking to get. There are two ways to solve for X in MATLAB. First, we can use the backslash operator (\) so that

$$X = A\backslash B$$

Second, we can solve for X as

$$X = A^{-1}B$$

which in MATLAB is the same as

$$X = inv(A)*B$$

We can also solve equations using the command **solve**. For example, given the quadratic equation $x^2 + 2x - 3 = 0$, we obtain the solution using the following MATLAB command:

```
>> [x]=solve('x^2 + 2*x - 3 =0')
 x =
  [-3]
[1]
```

indicating that the solutions are $x = -3$ and $x = 1$. Of course, we can use the command **solve** for a case involving two or more variables. We will see that in the following example.

Example C.1

Use MATLAB to solve the following simultaneous equations:

$$\begin{aligned} 25x_1 - 5x_2 - 20x_3 &= 50 \\ -5x_1 + 10x_2 - 4x_3 &= 0 \\ -5x_1 - 4x_2 + 9x_3 &= 0 \end{aligned}$$

SOLUTION

We can use MATLAB to solve this in two ways:

Method 1:
The given set of simultaneous equations could be written as

$$\begin{bmatrix} 25 & -5 & -20 \\ -5 & 10 & -4 \\ -5 & -4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad AX = B$$

We obtain matrix *A* and vector *B* and enter them in MATLAB as follows.

```
» A = [25  -5  -20; -5  10  -4; -5  -4  9]
       25  -5  -20
  A =  -5  10   -4
       -5  -4    9

» B = [50  0  0]'
       50
  B =   0
        0

» X = inv(A)*B
      29.6000
  X = 26.0000
      28.0000

» X = A\B
      29.6000
  X = 26.0000
      28.0000
```

Thus, $x_1 = 29.6$, $x_2 = 26$, and $x_3 = 28$.

Method 2:
Since the equations are not many in this case, we can use the command **solve** to obtain the solution of the simultaneous equations as follows:

```
[x1,x2,x3]=solve('25*x1 - 5*x2 - 20*x3=50', '-5*x1 + 10*x2
- 4*x3 =0', '-5*x1 - 4*x2 + 9*x3=0')
x1 =
148/5
x2 =
26
x3 =
28
```

which is the same as before.

Practice Problem C.1 Solve the problem with the following simultaneous equations using MATLAB:

$$3x_1 - x_2 - 2x_3 = 1$$
$$-x_1 + 6x_2 - 3x_3 = 0$$
$$-2x_1 - 3x_2 + 6x_3 = 6$$

Answer: $x_1 = 3 = x_3$, $x_2 = 2$.

C.5 PROGRAMMING HINTS

A good program should be well documented, of reasonable size, and capable of performing some computation with reasonable accuracy within a reasonable amount of time.

The following are some helpful hints that may make writing and running MATLAB programs easier:

- Use the minimum number of commands possible and avoid execution of extra commands. This is particularly true of loops.
- Use matrix operations directly as much as possible and avoid *for*, *do*, and/or *while* loops if possible.
- Make effective use of functions for executing a series of commands over several times in a program.
- When unsure about a command, take advantage of the help capabilities of the software.
- It takes much less time running a program using files on the hard disk than on a floppy disk.
- Start each file with comments to help you remember what it is all about later.
- When writing a long program, save frequently. If possible, avoid a long program; break it down into smaller subroutines.

C.6 OTHER USEFUL MATLAB COMMANDS

Some common useful MATLAB commands which may be used in this book are provided in Table C.8.

TABLE C.8
Other Useful MATLAB Commands

Command	Explanation
diary	Save screen display output in text format
mean	Mean value of a vector
min(max)	Minimum (maximum) of a vector
grid	Add a grid mark to the graphic window
poly	Convert a collection of roots into a polynomial
roots	Find the roots of a polynomial
sort	Sort the elements of a vector
sound	Play vector as sound
std	Standard deviation of a data collection
sum	Sum of elements of a vector

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Appendix D: Answers to Odd-Numbered Problems

Chapter 1

1.1 Proof

1.3 1/40

1.5 (a) energy signal, (b) neither power nor energy. (c) neither power nor energy.

1.7 (a) $x(t) = 10u(t) - 15u(t-2) + 5u(t-4)$

(b) $y(t) = 2r(t) - 2r(t-2) - 3u(t-2) + 3u(t-4)$

1.9 See Figure D.1.

1.11 $y(t) = 2r(t) - 2r(t-1) - 3u(t-1) - u(t-3)$

1.13 (a) $x(t) = 2u(t-1) - 2u(t-2)$

(b) $x(t) = 2\Pi[(t-1/2)/3]$

1.15 –0.2642

1.17 (a) $\delta(t-1)$

(b) $y(t) = 2r(t) - 2r(t-2) - 3u(t-2) + 3u(t-4)$

(c) $4\cos 4t\ u(t-3) - 0.5366\delta(t-3)$

1.19 (a) $x'(t) = \begin{cases} -1/2, & 0 < t < 1/2 \\ 1/2, & 1/2 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

(b) See Figure D.2.

1.21 See Figure D.3.

1.23 See Figure D.4.

1.25 $x[n] = 3\delta[n] + 4\delta[n-2] - 1.5\delta[n-3]$

1.27 See Figure D.5.

1.29 See Figure D.6.

1.31 See Figure D.7.

1.33 (a) $y(t) = 5\Lambda[(t-4)/2]$

(b) $y'(t) = 2.5[u(t-2) - 2u(t-4) + u(t-6)]$

$y''(t) = 2.5\delta(t-2) - 5\delta(t-4) + 2.5\delta(t-6)$

See Figure D.8.

1.35 See Figure D.9.

1.37 See Figure D.10.

1.39 See Figure D.11.

1.41 See Figure D.12.

1.43 See Figure D.13.

1.45 Proof

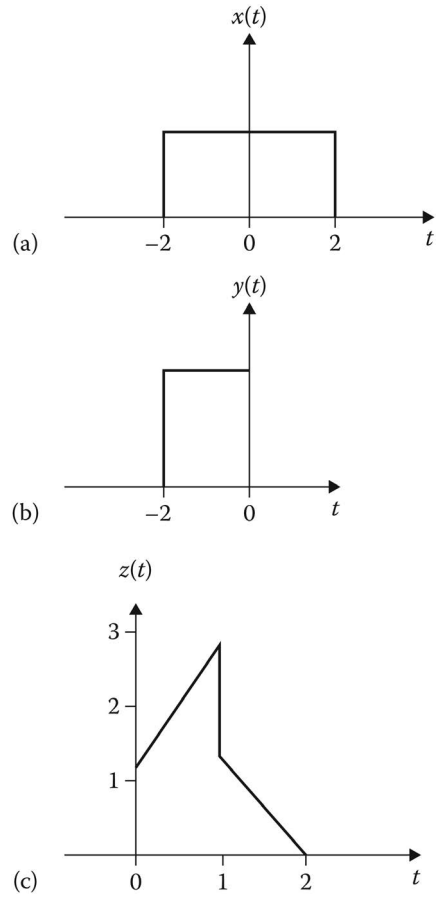


FIGURE D.1 For Prob. 1.9.

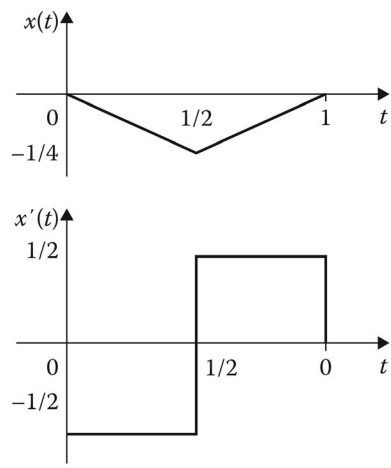


FIGURE D.2 For Prob. 1.19(b).

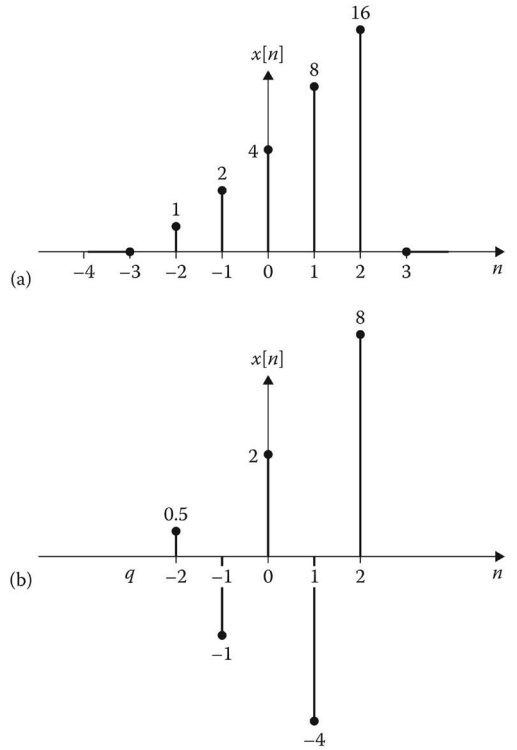


FIGURE D.3 For Prob. 1.21.

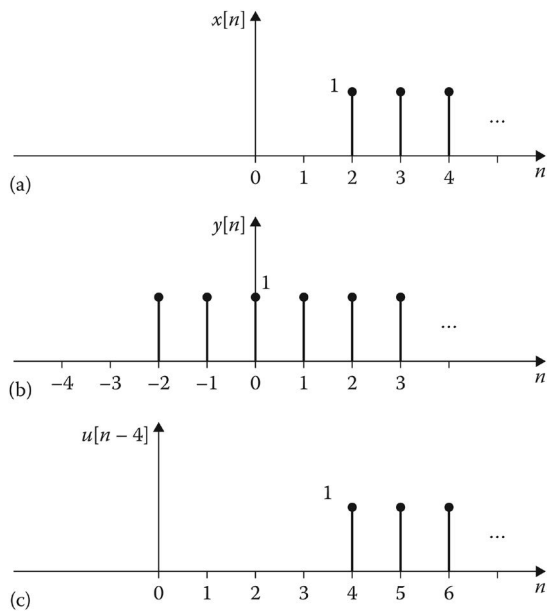


FIGURE D.4 For Prob. 1.23.

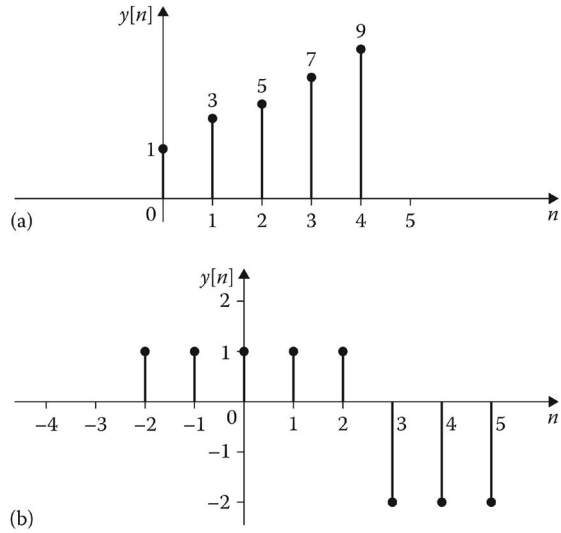


FIGURE D.5 For Prob. 1.27.

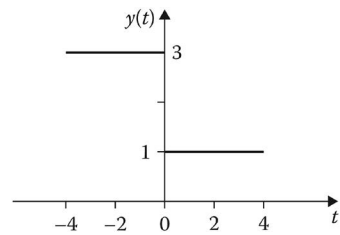


FIGURE D.6 For Prob. 1.29.

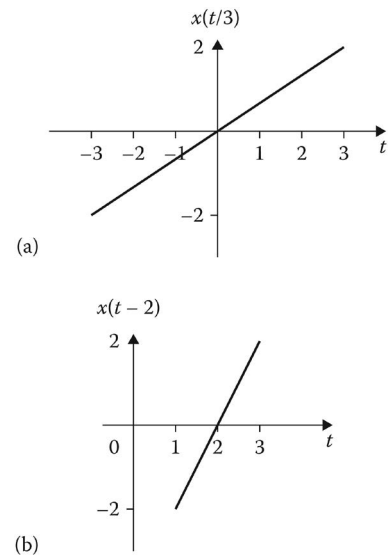


FIGURE D.7 For Prob. 1.31.

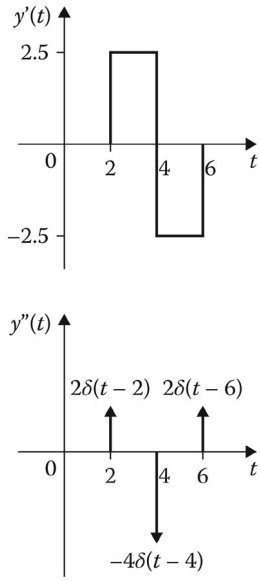


FIGURE D.8 For Prob. 1.33(b).

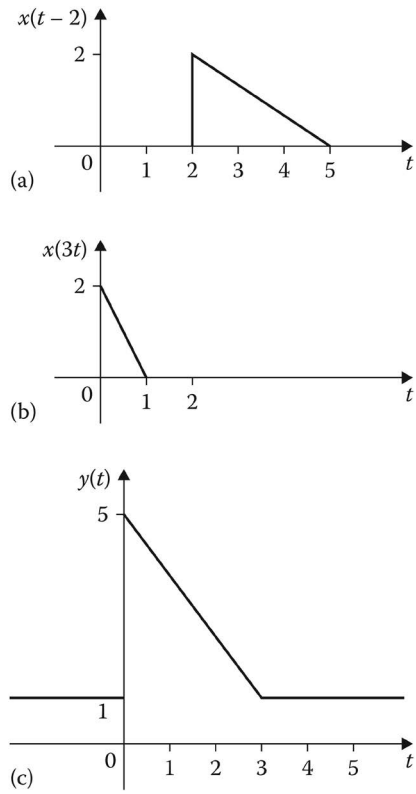


FIGURE D.9 For Prob. 1.35.

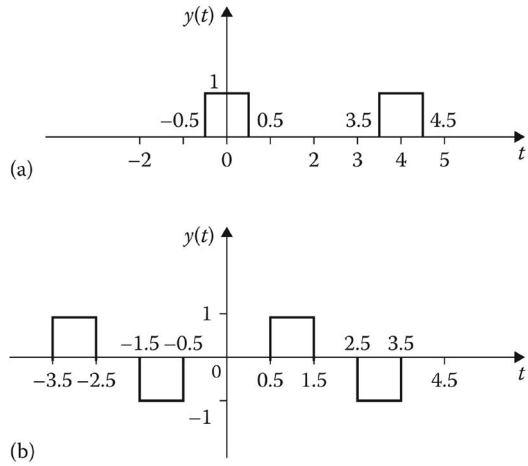


FIGURE D.10 For Prob. 1.37.

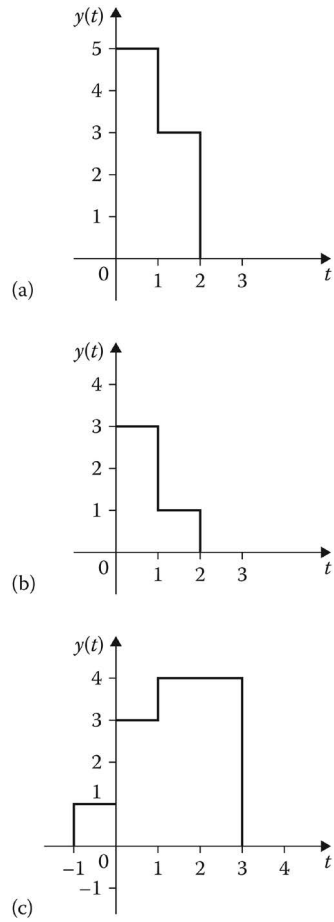


FIGURE D.11 For Prob. 1.39.

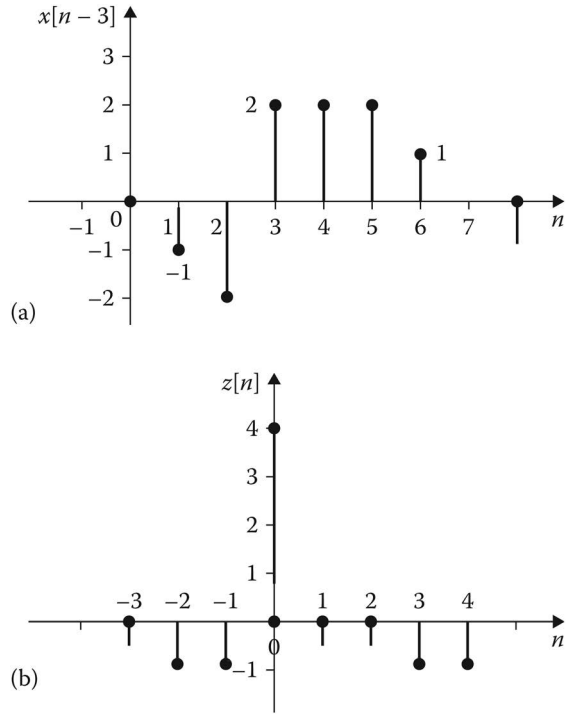


FIGURE D.12 For Prob. 1.41.

1.47 (a) Linear and time-varying system.
(b) Nonlinear and time-invariant system.
(c) Linear and time-invariant system.
(d) Nonlinear and time-varying system.

1.49 (a) Nonlinear, (b) Nonlinear, (c) Linear.

1.51 (a) Causal and memoryless, (b) Causal and memoryless.

1.53 (a) Not time-invariant, (b) Time-invariant.

1.55 $\dfrac{di(t)}{dt}+\dfrac{i(t)}{RC}=-\dfrac{1}{R}\dfrac{dv_s}{dt}$

1.57 $\dfrac{dy}{dt}+\dfrac{1}{RC}y=\dfrac{1}{RC}x$

1.59 See Figure D.14.

1.61 See Figure D.15.

Chapter 2

2.1 Proof

2.3 Proof

2.5 (a) $\dfrac{t}{a}(e^{at}-1)-\dfrac{1}{a^2}-\dfrac{e^{at}}{a^2}(at-1)$

(b) $0.5\cos(t)(t+0.5\sin(2t))-0.5\sin(t)(\cos(t)-1)$.

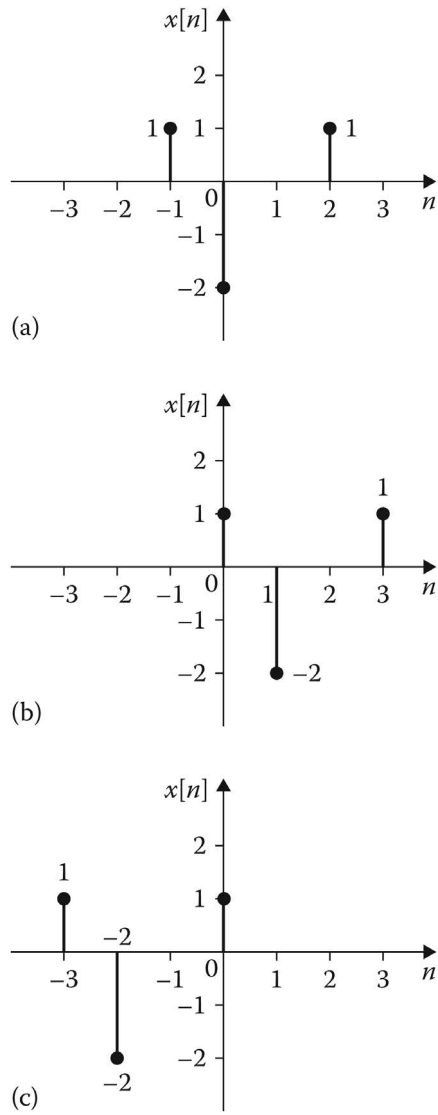


FIGURE D.13 For Prob. 1.43.

2.7 See Figure D.16.

2.9 $10(e^{-t} - e^{-2t})u(t)$

2.11 $-e^{-2t}u(t)$

2.13 (a) $\frac{1}{2}(1 - e^{-2t})u(t)$

(b) $(e^{-2t} - e^{-t})u(t)$

(c) $\frac{1}{4}\left(\cos 2t + \sin 2t - e^{-2t}\right)u(t)$

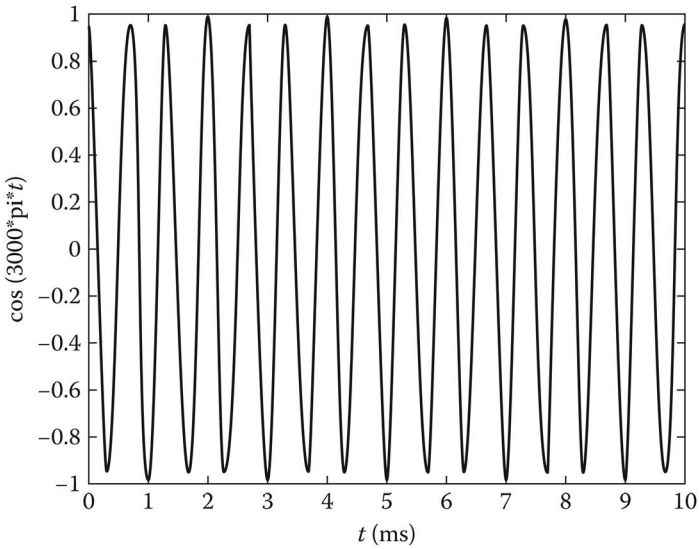


FIGURE D.14 For Prob. 1.59.

2.15 (a) unity system, (b) differentiator, (c) integrator

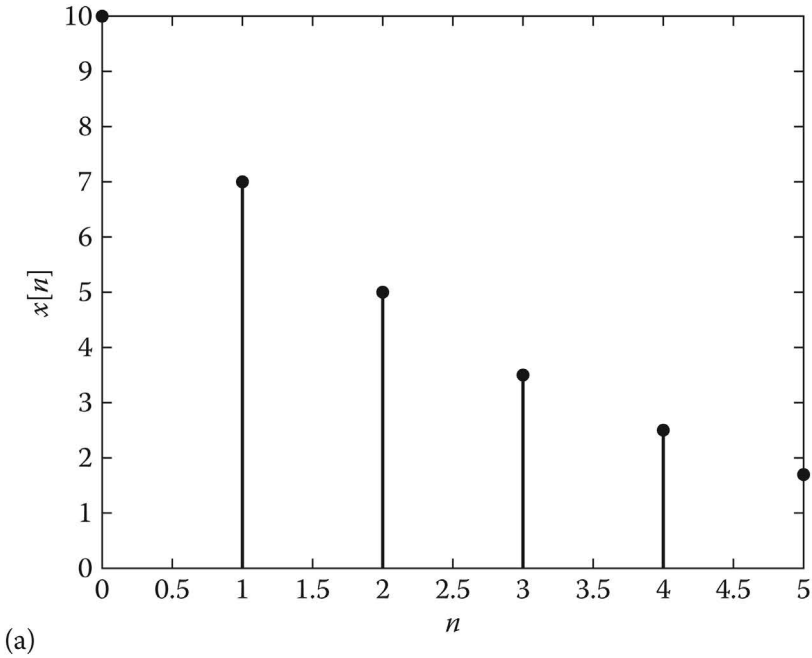
2.17 $$y(t)=\begin{cases} t, & 0<t<2 \\ 4-t, & 2<t<4 \\ 0, & \text{otherwise} \end{cases}$$

2.19 $4\left(e^{-t}-e^{-2t}\right)u(t)$

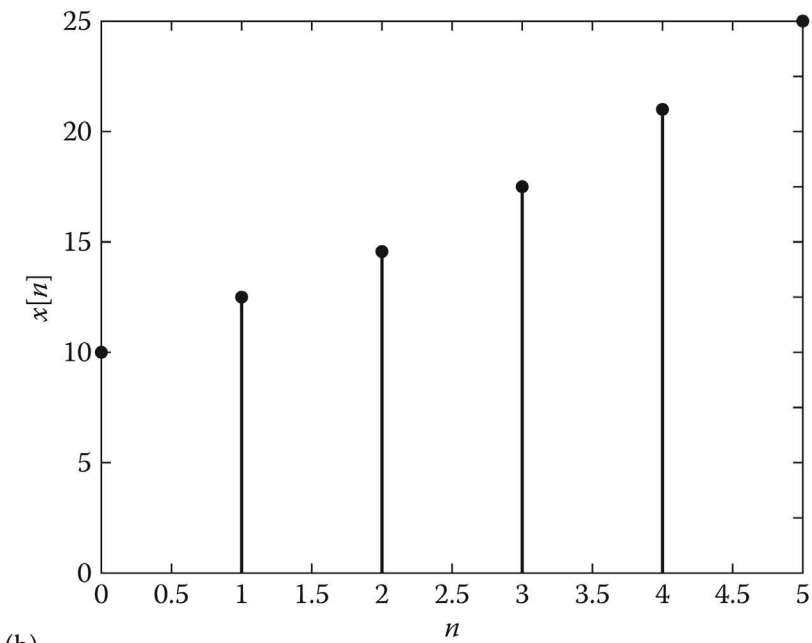
2.21 $$y(t)=\begin{cases} 4(t+1), & -1<t<0 \\ 4, & 0<t<1 \\ 4(2-t), & 1<t<2 \\ 0, & \text{otherwise} \end{cases}$$

2.23 (a) $$y(t)=\begin{cases} t-1, & 1<t<2 \\ 1, & 2<t<3 \\ 4-t, & 3<t<4 \\ 0, & \text{otherwise} \end{cases}$$

(b) $$y(t)=\begin{cases} 0, & t<-1 \\ 2(1-e^{-2t-2}), & -1\le t\le 0 \\ 2(2e^{-2t}-e^{-2t-4}-1), & 0\le t\le 1 \\ 2(2e^{-2t}-e^{-2t-4}-e^{-2t+2}), & t\ge 1 \end{cases}$$



(a)



(b)

FIGURE D.15 For Prob. 1.61.

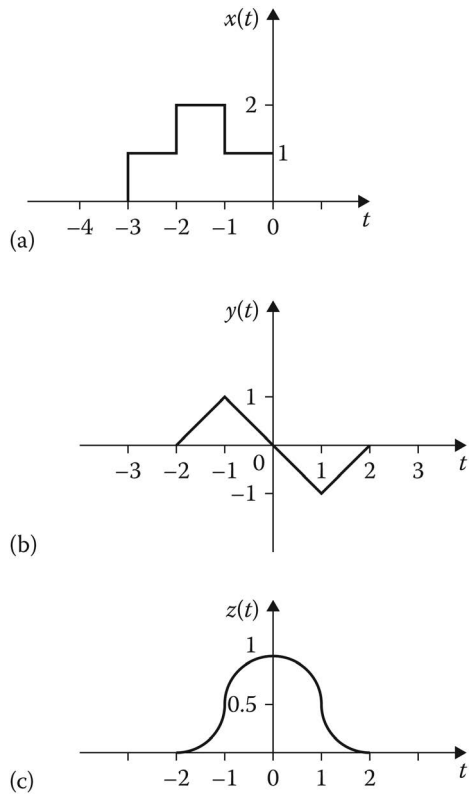


FIGURE D.16 For Prob. 2.7.

2.25 $$x(t) = \begin{cases} t^2 + 2t + 1, & -1 < t < 0 \\ -2t^2 + 2t + 1, & 0 < t < 1 \\ t^2 - 4t + 4, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

2.27 $$z(t) = \begin{cases} t^2 - 2t + 1, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ -t^2 + 6t - 8, & 3 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

2.29 $$z(t) = \begin{cases} t^2 - 2t + 1, & 1 < t < 2 \\ -2t^2 + 8t - 7, & 2 < t < 3 \\ t^2 - 6t + 8, & 3 < t < 4 \\ 0, & \text{otherwise} \end{cases}$$

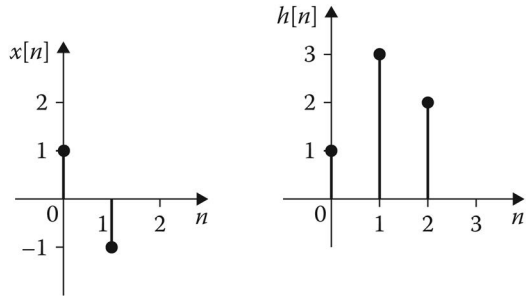


FIGURE D.17 For Prob. 2.39(a).

2.31 $h(t) = 8(e^{-t} - e^{-2t})u(t)$

2.33 $h(t) = h_5(t) * \left[h_4(t) + h_2(t) * h_3(t) - h_1(t)\right]$

2.35 $y[2] = 0.48$, $y[5] = 0.05632$

2.37 $y[n] = 6\left[3^{n+1} - 2^{n+1}\right]$

2.39 (a) See Figure D.17.

(b) $$y[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ -1, & n = 2 \\ -2, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

2.41 $y[n] = 2\big(\delta[n] + \delta[n-2] + \delta[n-4]\big)$
which is sketched in Figure D.18.

2.43 $$z[n] = \begin{cases} 2, & n = 0 \\ 1, & n = 1 \\ 1, & n = 2 \\ -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

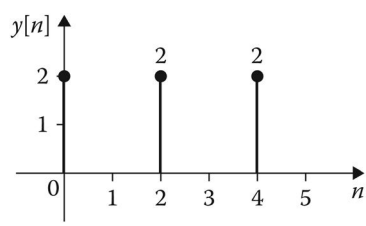


FIGURE D.18 For Prob. 3.41.

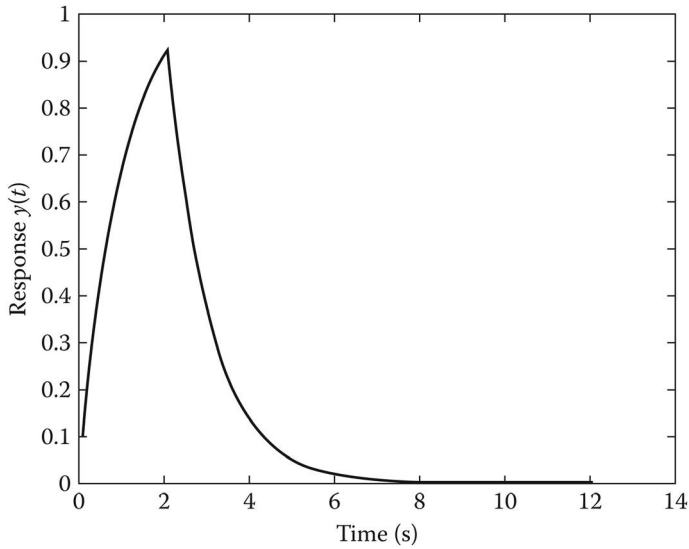


FIGURE D.19 For Prob. 2.51(a).

2.45 $y[n]=\frac{2}{3}(0.4)^n(2.5^{n+1}-1)-(0.5)^{n+1}(2^{n+1}-1)$

2.47 (a) $y[n]=(n+2)(0.4)^n u[n]+1.25(0.4)^n u[n-1]$

(b) $y[n]=(n+1)(0.4)^n u[n]+n(0.4)^{n-1}u[n-1]$

2.49 [2 3]

2.51 (a) For $y(t)$, see Figure D.19.

(b) $y(t)=\begin{cases}1-e^{-t}, & 0<t<2\\ e^{-t}(e^2-1), & t>2\end{cases}$

2.53 See Figure D.20.

2.55 h = [2 8 8 –16]

2.57 Stable

2.59 Not BIBO stable

2.61 $v_o(t)=2(1-e^{-t})u(t)$

Chapter 3

3.1 $X(s)=\frac{1}{s}\left(2-3e^{-2s}+e^{-3s}\right)$

3.3 $F(s)=\frac{2}{s^2}(1-e^{-s}-se^{-2s})$

3.5 (a) $X_1(s)=\frac{1}{(s-2)^2}$

(b) $X_2(s)=\frac{s^2-16}{(s^2+16)^2}$

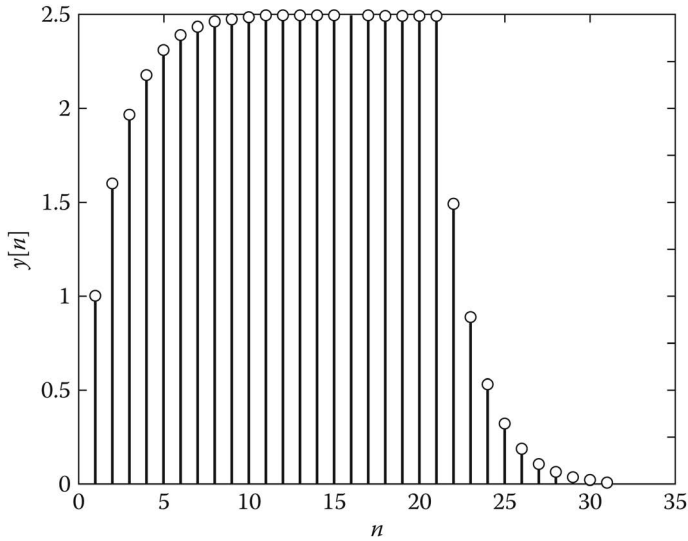


FIGURE D.20 For Prob. 2.53.

(c) $X_3(s) = e^{2s}$

(d) $X_4(s) = \dfrac{5}{\sqrt{2}}\left(\dfrac{s-2}{s^2+4}\right)$

(e) $X_5(s) = e^{-s}\left(\dfrac{2}{s^3} - \dfrac{2}{s^2} + \dfrac{1}{s}\right)$

3.7 $X(s) = \dfrac{1}{s}\left[2 - e^{-s} + e^{-2s}\right]$

3.9 $X(s) = \dfrac{1-e^{-(s+2)}}{s+2}, Y(s) = \dfrac{\pi(1+e^{-s})}{s^2+\pi^2}$

3.11 $y(\infty) = 2$

3.13 $f(0) = 0, f(\infty) = 3$

3.15 0.5

3.17 10

3.19 (a) $g(t) = \left(\dfrac{2}{3} - \dfrac{1}{2}e^{-t} - \dfrac{1}{6}e^{-3t}\right)u(t)$

(b) $h(t) = \dfrac{5}{27}(t-2)^2 e^{-5(t-2)/3}u(t-2)$

3.21 $y(t) = 8(t-1)u(t-1) - 4(t-2)y(t-2) - 2u(t-2) - 6u(t-3)$

3.23 (a) $x(t) = \delta'(t) + \delta(t) + e^{-t}u(t)$

(b) $y(t) = \delta(t) + \left(2 + t - e^{-t}\right)u(t)$

3.25 (a) $f(0) = 5, f(\infty) = 0$

(b) $f(t) = -5e^{-2t} + 10e^{-3t}$

3.27 $h(t) = te^{-t}u(t)$

3.29 $y(t) = \left(\frac{5}{3} - 5e^{-2t} + \frac{10}{3}e^{-3t}\right)u(t)$

3.31 $H = \frac{1}{s^2 + 2as + a^2 + b}$

3.33 $H = \frac{G_2G_3(G_1 + F_2)}{1 + G_2F_1}$

3.35 $H = \frac{H_1}{(1 + H_1H_2 - H_3)}$

3.37 $y(t) = 5\left(1 - 2e^{-t} + e^{-2t}\right)u(t)$

3.39 (a) $h(t) = 3e^{-2t} - 2e^{-3t}, t > 0$

(b) $y(t) = \frac{5}{6} - 1.5e^{-2t} + \frac{2}{3}e^{-3t}, t > 0$

(c) $y(t) = \frac{-19}{36} + \frac{5t}{6} + \frac{3}{4}e^{-2t} - \frac{2}{9}e^{-3t}, t > 0$

3.41 $y(t) = (0.1765e^{-t} + 0.7059te^{-t} - 0.1765\cos 2t + 0.2353\sin 2t)u(t)$

3.43 $H(s) = 1 - \frac{1.41s + 1}{s^2 + 1.41s + 1}$

3.45 (a) $H(s) = \frac{s - 3}{s^2 + 2s + 2}$

(b) $h(t) = \left(e^{-t}\cos t - 4e^{-t}\sin t\right)u(t)$

3.47 $y(t) = -\frac{16}{5}e^{-2t} + \frac{16}{5}e^{-0.5t}\cos(0.5t) - \frac{8}{5}e^{-0.5t}\sin(0.5t), t > 0$

3.49 $v_o(t) = -20e^{-2t} + 50e^{-5t}, t > 0$

3.51 $H(s) = \frac{s^2 + 2s + 1}{s^2 + 3s + 1}$

3.53 $I_1 = \frac{2(2s + 1)}{(s + 1)(s^2 + 3s + 1)}, I_2 = \frac{2s}{(s + 1)(s^2 + 3s + 1)}$

3.55 (a) unstable, (b) stable, (c) unstable

3.57 (a) $X =$

```
2/s^2 – 4/s + 2/s^3 + 6/s^4
```

(b) $X =$

```
(2*s^2)/(s^2 + 9)^2 – 1/(s^2 + 9)
```

(c) $X =$

```
2/((s + 1)^2 + 4)
```

3.59 $x(t) = 0.3125e^{-5t} + 0.6875e^{-t} + 1.25te^{-t}, t > 0$

3.61 (a) $x(t) = 3\delta(t) - 14e^{-4t} - e^{-3t}$
(b) and (c) involve complex algebra.

3.63 (a) $z =$
2
$p =$

–1.0000 + 3.0000i
–1.0000 – 3.0000i

(b) $z =$

–1.0000 + 2.0000i
–1.0000 – 2.0000i

$p =$

0.0000 + 0.0000i
–2.0000 + 3.0000i
–2.0000 – 3.0000i

(c) $z =$

–9.4721
–0.5279

$p =$

–1.5956 + 2.2075i
–1.5956 – 2.2075i
–0.8087 + 0.0000i

3.65 See Figure D.21

Chapter 4

4.1 $x(t) = 7.5 - \sum_{n=odd}^{\infty} \frac{10}{n\pi} \sin nt$

4.3 $x(t) = \frac{20}{\pi} \sum_{\substack{n=1 \\ n=odd}}^{\infty} \frac{(-1)^n}{n} \sin(n\pi/2)\cos(n\pi t/2)$

4.5 $x(t) = 5 + \frac{10}{\pi} \sum_{k=1}^{\infty} \frac{1}{n}(1 - \cos n\pi)\sin nt$

4.7 $y(t) = 0.5 + \sum_{n=1}^{\infty} \left[\frac{2}{n\pi} \sin(\pi n/2)\cos(nt/2) + \frac{2}{n\pi}(1 - \cos(n\pi/2))\sin(nt/2) \right]$

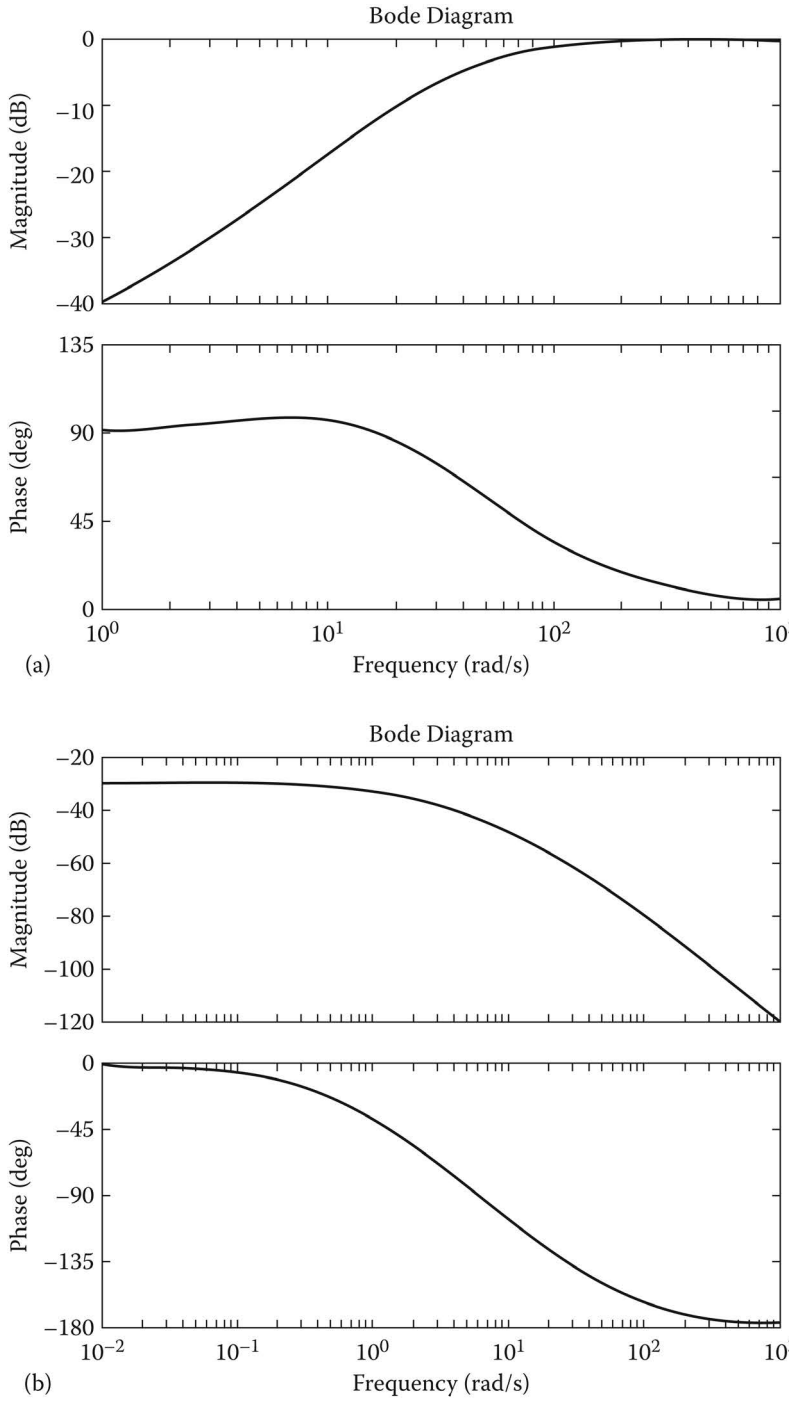


FIGURE D.21 For Prob. 3.65.

4.9 $y(t) = \sum_{n=1}^{\infty} \left[a_n \cos(n\pi t/2) + b_n \sin(n\pi t/2) \right],$

where $a_n = \frac{4}{n\pi} \sin(n\pi/2), b_n = \frac{2}{n\pi} \left[1 - 2\cos(n\pi/2) + \cos n\pi \right]$

4.11 $g(t) = 5 + \sum_{n=1}^{\infty} \frac{20}{n^2\pi^2} (\cos n\pi - 1) \cos nt$

4.13 $x(t) = 2.5 + \sum_{n=1}^{\infty} a_n \cos(n\pi t/2) + b_n \sin(n\pi t/2)$

where $a_n = \frac{20}{n^2\pi^2} \left[5\cos n\pi/2 - \cos n\pi - 1 \right], b_n = \frac{40}{n^2\pi^2} \sin n\pi/2$

4.15 Proof

4.17 $x(t) = \sum_{n=-\infty}^{\infty} \frac{2}{jn\pi} (\cos n\pi - 1) e^{jnt}$

4.19 $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnt}, c_n = \frac{2}{a^2 n^2} \left[(1 + jn)(-1)^n - 1 \right]$

4.21 $z(t) = \sum_{n=-\infty}^{\infty} \frac{j5}{n\pi} (\cos n\pi - 1) e^{jn\pi t}$

4.23 $c_n = \frac{\left[e^{(1-jn\pi)} - 1 \right]}{2(1 - jn\pi)}$

4.25 (a) 100, (b) 0, (c) 1

4.27 (a) $T = 6.283s, \omega_o = 1$ rad/s

(b) $x(t) = 0.4 + 0.4\cos t + 0.24\cos 2t + 0.16\cos 3t$

4.29 $10\cos 3t = 5e^{-j3t} + 5e^{j3t}$

4.31 (a) $\beta_n = c_n$

(b) $\alpha_n = c_{-n}$

(c) $\gamma_n = \begin{cases} c_0 + 1, & n = 0 \\ c_n, & n \neq 0 \end{cases}$

4.33 Proof

4.35 (a) Will have a DC component, no odd harmonic, only cosine terms.
(b) Will have a DC component, no odd harmonic, only cosine terms.
(c) Will have a DC component, no odd harmonic, only cosine terms.

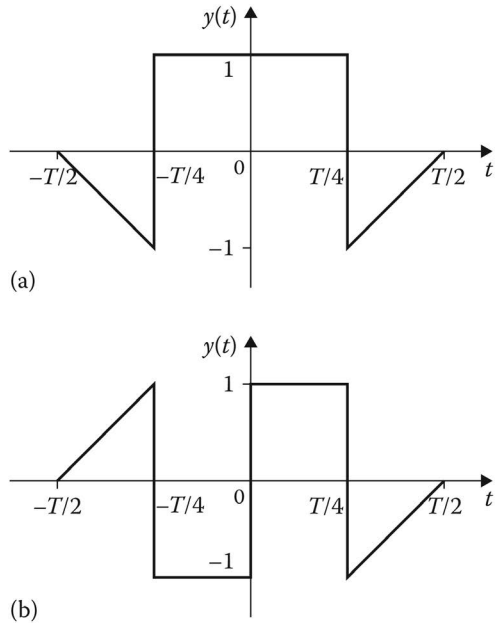


FIGURE D.22 For Prob. 4.37.

(d) No DC component, only odd harmonics, only sine terms.
(e) No DC component, only odd harmonics, only sine terms.
(f) No DC component, only odd harmonics, only sine terms.
(g) No DC component, both even and odd harmonics, both sine and cosine terms.
(h) Will have a DC component, no odd harmonic, only cosine terms.

4.37 See Figure D.22.

4.39 247.5 W

4.41 $d_n = \dfrac{jV_o \sin^2(n\pi/2)}{n\pi/4}$

4.43 $v(t) = 0.931\cos(2\pi t - 41.29°) + 0.316\cos(4\pi t - 230.9°)$

4.45 $v_o(t) = \displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{9n^2\pi^2 + 1}} \sin\left[2n\pi t - \tan^{-1}\left(\frac{1}{3n\pi}\right)\right]$

4.47 $v_o(t) = \displaystyle\sum_{n=1}^{\infty} \frac{2(1-\cos \pi n)}{\sqrt{1+n^2\pi^2}} \cos(n\pi t - \tan^{-1} n\pi)$

4.49 $i(t) = \dfrac{4}{\pi} \displaystyle\sum_{n=odd}^{\infty} \frac{1}{\sqrt{n^2+4}} \sin\left[nt - \tan^{-1}(-2/n)\right]$

4.51 See Figure D.23.

4.53 See Figure D.24.

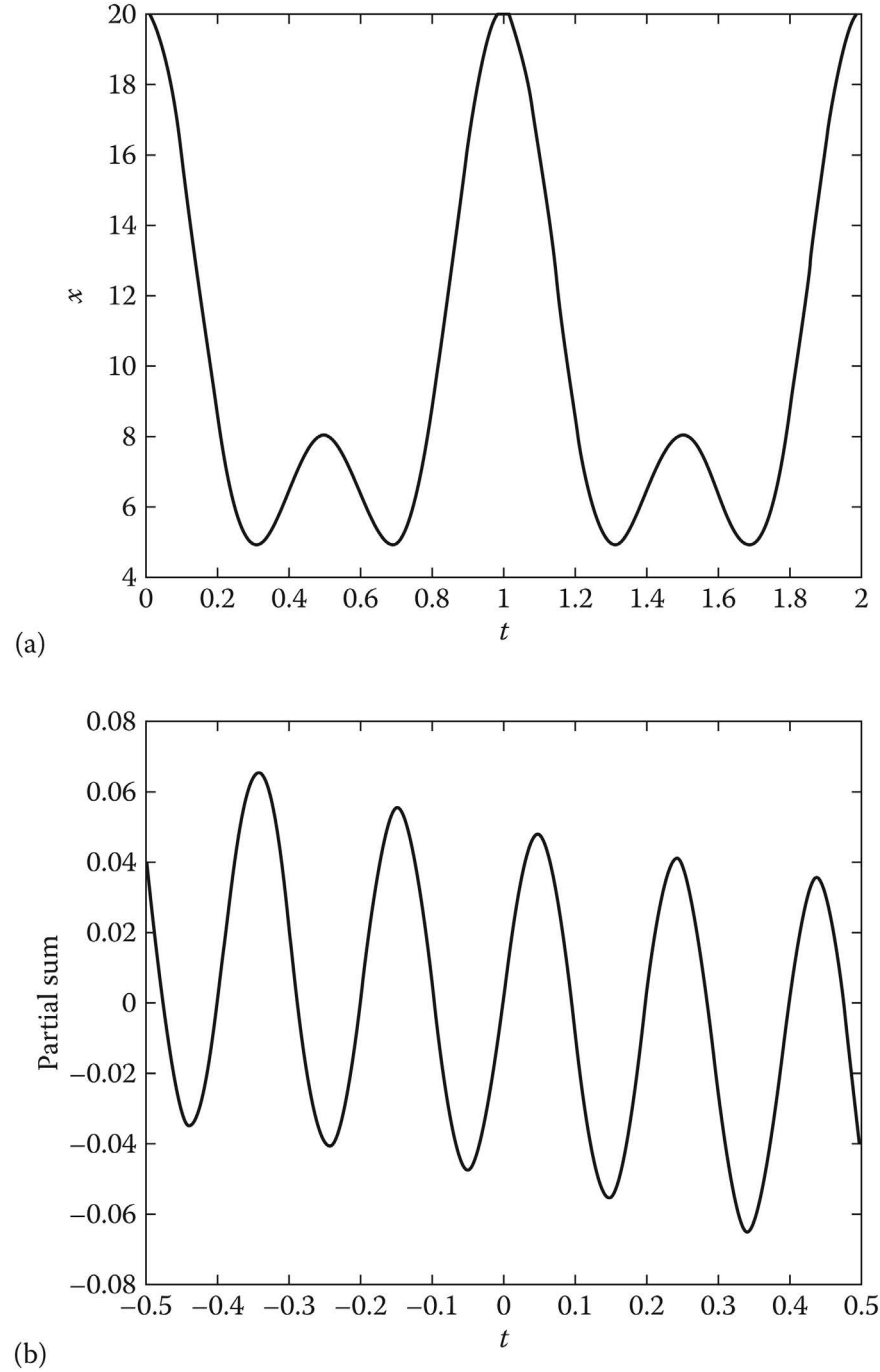


FIGURE D.23 For Prob. 4.51.

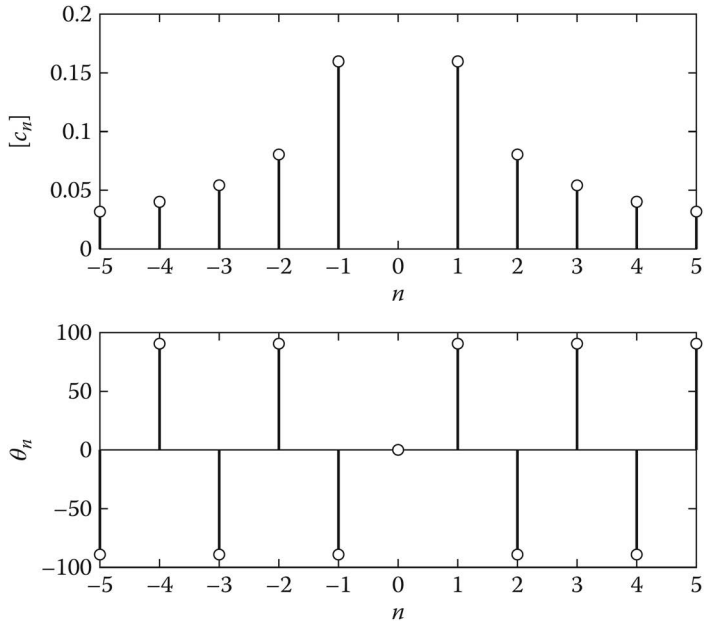


FIGURE D.24 For Prob. 4.53.

Chapter 5

5.1 $X(\omega) = \dfrac{10}{\omega}\sin 2\omega$

5.3 $Z(\omega) = \dfrac{j}{\omega}(e^{-j3\omega} + e^{-j\omega} - 2)$

5.5 $X(\omega) = \dfrac{-2}{j\omega}$

5.7 (a) $X(\omega) = \dfrac{1}{j\omega}(1 - 2e^{-j\omega} + e^{-j2\omega})$

(b) $Y(\omega) = \dfrac{2}{j\omega}(\cos\omega - 1)$

5.9 (a) $X(\omega) = 1 - e^{-j2\omega}$

(b) $Y(\omega) = \dfrac{2}{\omega}\sin\omega$

5.11 Proof

5.13 (a) $X(\omega) = 2\pi[\delta(\omega + \omega_o)e^{-j\theta} + \delta(\omega - \omega_o)e^{j\theta}]$

(b) $Y(\omega) = 2\pi[\delta(\omega + 100)e^{j\pi/4} + \delta(\omega - 100\pi)e^{-j\pi/4}]$

5.15 (a) $Y(\omega)=\dfrac{1}{4}e^{-j2.5\omega}\dfrac{(10+j\omega)}{-0.25\omega^2+j\omega+1}$

(b) $Y(\omega)=\dfrac{j(j\omega^2+10\omega+j2\omega-j9)}{(-\omega^2+j2\omega+1)^2}$

(c) $Y(\omega)=\dfrac{j}{2}\left[\dfrac{5+j(\omega+2)}{-(\omega+2)^2+j2(\omega+2)+1}+\dfrac{5+j(\omega-2)}{-(\omega-2)^2+j2(\omega-2)+1}\right]$

(d) $Y(\omega)=\dfrac{-\omega^2(5+j\omega)}{-\omega^2+j2\omega+1}$

5.17 (a) $Y(\omega)=\dfrac{j\omega e^{-j2\omega}}{4+j\omega}$

(b) $Z(\omega)=\dfrac{j\omega}{3(12+j\omega)}$

(c) $H(\omega)=\dfrac{-\omega^2}{4+j\omega}$

5.19 (a) $\mathrm{X}(\omega)=4\pi\left[\delta(\omega+2)-\delta(\omega-2)\right]+2$

(b) $Y(\omega)=j4\pi\left[\delta(\omega+2)-\delta(\omega-2)\right](1-\cos\omega)$

5.21 $Y(\omega)=\dfrac{j}{2}\left[\pi\delta(\omega+2)-\pi\delta(\omega-2)+\dfrac{1}{j(\omega+2)}+\dfrac{1}{j(\omega-2)}\right]$

5.23 (a) $X(\omega)=2(\cos\omega+\cos 2\omega)$

(b) $Y(\omega)=\dfrac{4}{(1+j\omega)^2}$

(c) $Z(\omega)=\dfrac{4(2)}{(1+j\omega)^2+4^2}=\dfrac{8}{(1+j\omega)^2+16}$

5.25 Proof

5.27 Proof

5.29 (a) Proof, (b) Examples are $\cos t$ and $e^{-a|t|}$

5.31 (a) 60, (b) 20π, (c) 2000π

5.33 (a) $\dfrac{2}{(a+j\omega)^2}$

(b) $\dfrac{n!}{(a+j\omega)^{n+1}}$

5.35 (a) $x(t)=2\delta(t)-2e^{-t}u(t)$

(b) $y(t)=\dfrac{10}{\pi(t^2+1)}$

(c) $z(t)-\left(-2e^{-2t}+3e^{-3t}\right)u(t)$

5.37 $x(t) = \frac{5}{\pi j}[-\cos t + \cos 2t]$

5.39 (a) $H(\omega) = \frac{2 + j\omega}{1 + j\omega}$

(b) $y(t) = \frac{1}{4}(-j6\sin t + j2\cos t)$

(c) $y(t) = \frac{1}{2}\left(e^{-t} + e^{-3t}\right)u(t)$

5.41 $v_o(t) = 12\left(e^{-2t} - e^{-3t}\right)u(t)$

5.43 $i_o(t) = 10\left(e^{-2t} - e^{-3t}\right)u(t)$

5.45 $v_o(t) = 16(e^{-t} - e^{-2t})u(t)$

5.47 $v_o(t) = \left(2e^{-2t} - 2e^{-4t}\right)u(t)$

5.49 $y(t) = \left(e^{-2t} - e^{-3t}\right)u(t)$

5.51 $y(t) = \left(e^{-t} - e^{-2t}\right)u(t)$

5.53 $Y(\omega) = 2X(t) + \frac{1}{2}X(\omega + \omega_o) + \frac{1}{2}X(\omega - \omega_o)$

5.55 (a) 60 Hz, (b) 50 Hz

5.57 66.67 J

5.59 2010.62 J

5.61 $X =$

$$\frac{5}{wi + 2 - 4i} + \frac{5}{wi + 2 + 4i}$$

The plot of $|X(\omega)|$ is in Figure D.25.

5.63 See Figure D.26.

5.65 See Figure D.27.

Chapter 6

6.1 (a) Proof, (b) Proof

6.3 $j2[\sin\Omega + \sin 2\Omega + \sin 3\Omega]$

6.5 $X(\Omega) = 8\cos 2\Omega - 2\cos\Omega$

6.7 $Y(\Omega) = 2e^{j2\Omega} - 3e^{j\Omega} + 1 + 4e^{-j2\Omega}$

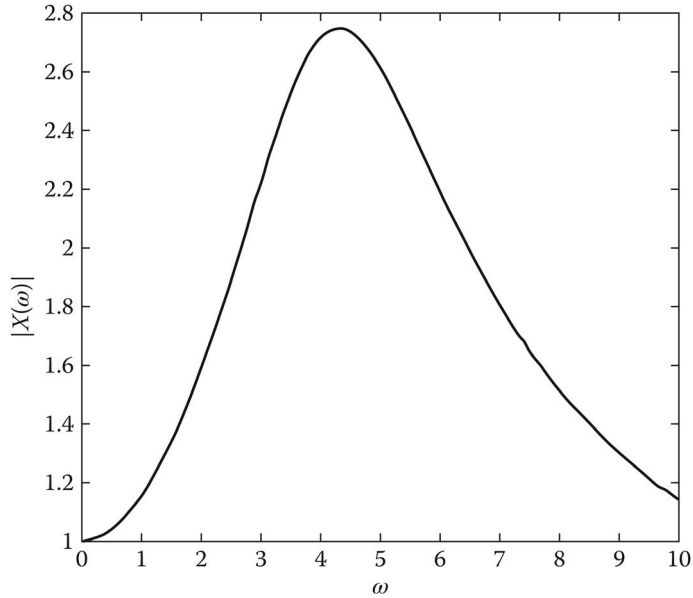


FIGURE D.25 For Prob. 5.61.

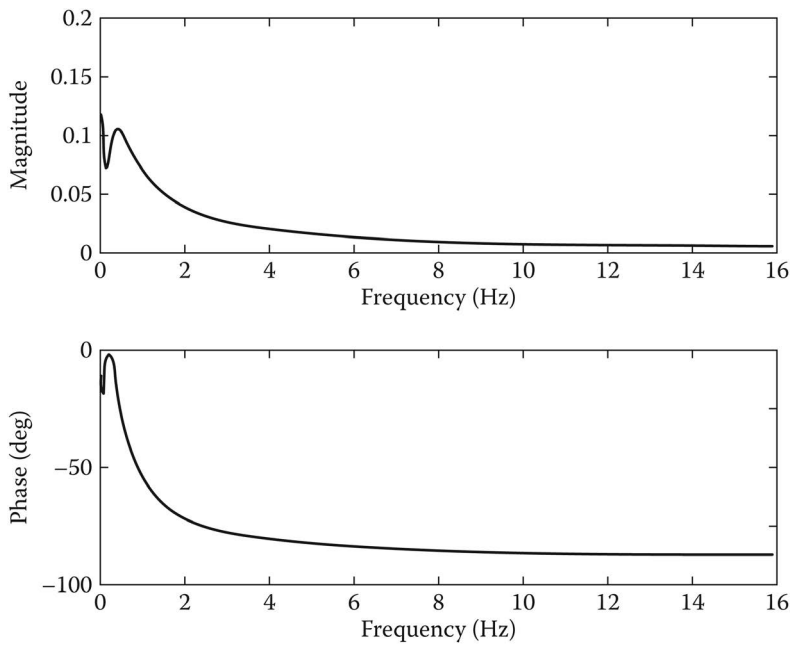


FIGURE D.26 For Prob. 5.63.

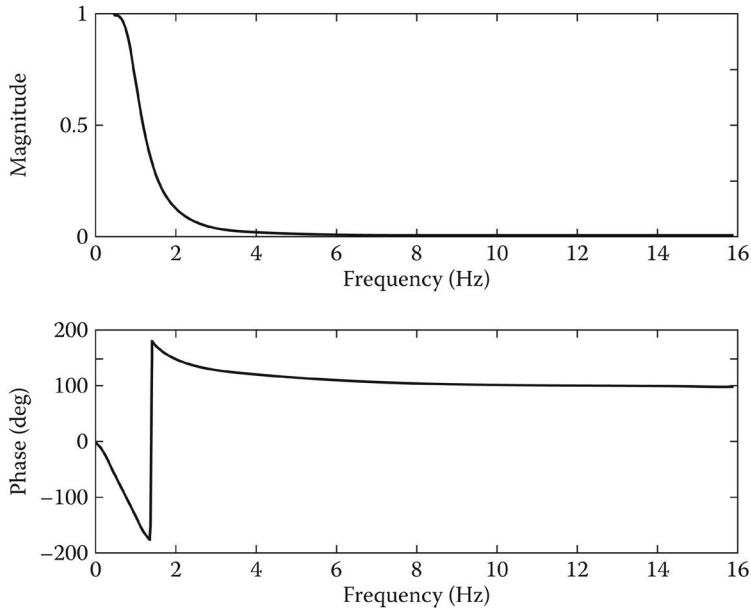


FIGURE D.27 For Prob. 5.65.

6.9 (a) $\dfrac{1}{1-2e^{j\Omega}}$

(b) $j0.5\sin 2\Omega$

(c) $\dfrac{e^{j2\Omega}}{1-0.8e^{-j\Omega}}$

6.11 $X(\Omega)=\dfrac{-2ja\sin\Omega}{1+a^2-2a\cos\Omega}$

6.13 (a) $Y(\Omega)=\dfrac{2}{3+e^{j\Omega}}$

(b) $Z(\Omega)=\dfrac{-2e^{-j\Omega}}{(3+e^{-j\Omega})^2}$

(c) $W(\Omega)=\dfrac{2(1+e^{-j\Omega})}{3+e^{-j\Omega}}$

(d) $V(\Omega)=\dfrac{2}{3-e^{-j\Omega}}$

6.15 (a) $x[n]=\dfrac{5}{j}\big[\delta[n+4]-\delta[n-4]\big]$

(b) $x[n]=\dfrac{1}{2}\big[\delta[n+2]+2\delta[n]+\delta[n-2]\big]$

6.17 $x[n] = \begin{cases} 3, & n = -2 \\ 0, & n = -1 \\ 4, & n = 0 \\ -2, & n = 1 \\ 0, & n = 2 \\ 1, & n = 3 \\ 0, & n = 4 \\ 5, & n = 5 \end{cases}$

6.19 $X(\Omega) = \sin c\left(\frac{n\pi}{2}\right)$

6.21 (a) $x[n] = \frac{1}{n\pi}(3\sin n + \sin 2n)$

(b) $x[n] = \frac{\cos(n\pi/2)}{\pi(1-n^2)}$

6.23 Proof

6.25 (a) $y[n] = 4/3$

(b) $y[n] = \left(\frac{1}{3}\right)^n [u[n] + u[n-1]]$

(c) $y[n] = 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$

6.27 Proof

6.29 (a) $X[k] = [6\ -2 + j2\ -2\ -2 - j2]$

(b) $X[k] = [0\ 0\ 4 - j4\ 0\ 0\ 0\ 4 + j4\ 0]$

6.31 Proof

6.33 $x[0] = 1$, $x[1] = -0.5257$, $x[2] = -0.8507$, $x[3] = 0.8507$, $x[4] = 0.5257$

6.35 22

6.37 Proof

6.39 Proof

6.41 See Figure D.28.

6.43 $X =$

Columns 1 through 7

10.0000 7.2426 + 7.8284i −6.0000 −1.2426 − 2.1716i −2.0000 −1.2426 + 2.1716i −6.0000

Column 8

7.2426 − 7.8284i

6.45 See Figure D.29.

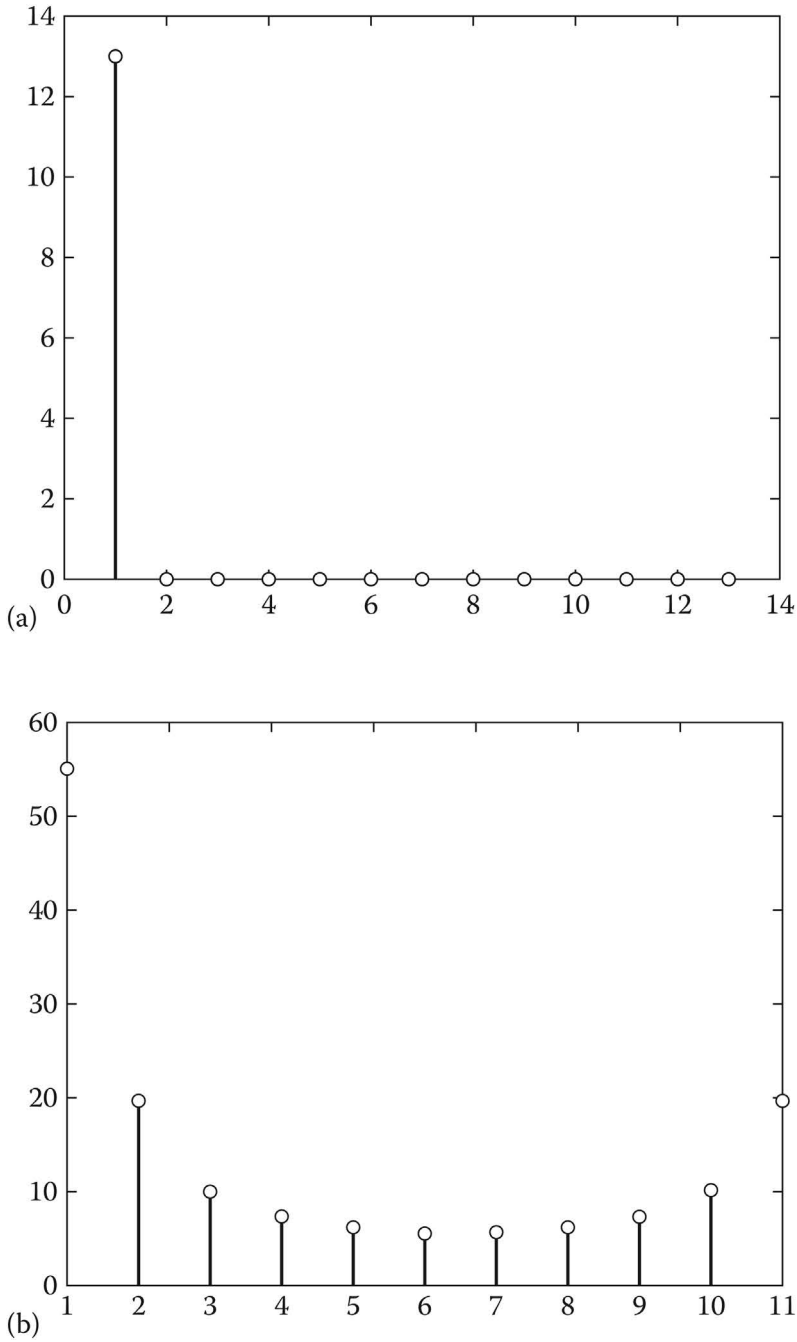
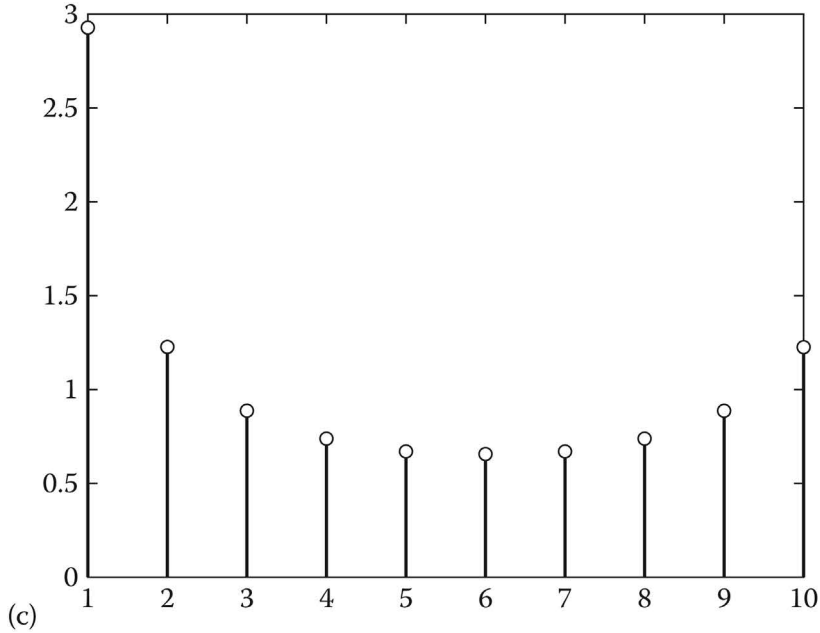
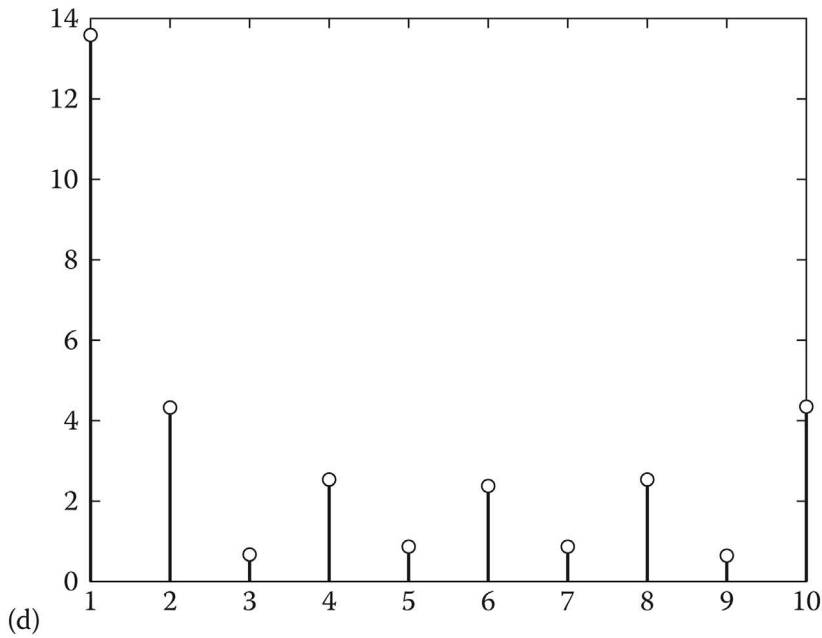


FIGURE D.28 For Prob. 6.41. (*Continued*)



(c)



(d)

FIGURE D.28 (*Continued*)

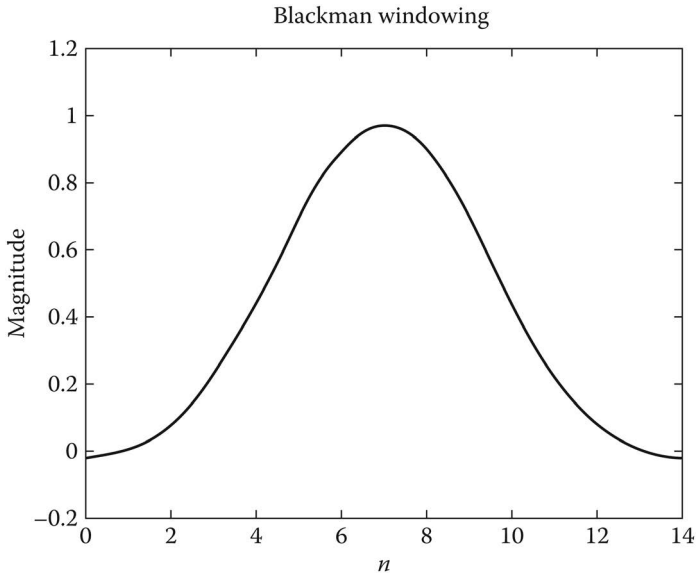


FIGURE D.29 For Prob. 6.45.

Chapter 7

7.1 (a) $1+z^{-1}-z^{-2}-z^{-3}$, (b) $1-z^{-1}+z^{-2}-z^{-3}$

7.3 $Y(z)=\dfrac{1}{z}+\dfrac{2}{z^2}+\dfrac{3}{z^3}+\dfrac{4}{z^4}$

7.5 $\dfrac{6z}{3z-2}-\dfrac{5z}{5z-2}, |z|>2/3$

7.7 (a) $X(z)=\dfrac{z}{z^m(z-1)}$

(b) $X(z)=\dfrac{az}{(z-a)^2}$

(c) $X(z)=\dfrac{z^2+za}{z^2+2za+a^2}$

7.9 (a) $X(z)=\dfrac{2z}{2z-1}+\dfrac{3z}{3z+1}$

(b) $X(z)=\dfrac{4z}{4z-1}$

(c) $X(z)=1+\dfrac{1}{2z}+\dfrac{1}{(2z)^2}+\dfrac{1}{(2z)^3}+\cdots$

7.11 (a) $X(z) = \dfrac{9z}{3z-2} + \dfrac{12z}{3z+1}$

(b) $Y(z) = \dfrac{z(2-e^{-a}-z)}{(z-1)(1-e^{-a})}$

7.13 Proof

7.15 $A = 4, \Omega = 36.87^{\circ}$

7.17 (a) $x[0] = 0$, (b) $y[0] = 1$, (c) $h[0] = 0$

7.19 (a) $x[\infty] = 1.5$, (b) $x[\infty] = 1.5$

7.21 (a) $x[\infty] = 0$, (b) $x[\infty] = 0$

7.23 (a) $Y(z) = \dfrac{2}{(z^2+3z+1)}$

(b) $Y(z) = \dfrac{j}{2}\left[\dfrac{2e^{j\pi/4}z}{e^{j\pi/2}z^2+3e^{j\pi/4}z+1} - \dfrac{2e^{-j\pi/4}z}{e^{-j\pi/2}z^2+3e^{-j\pi/4}z+1}\right]$

(c) $Y(z) = \dfrac{2z(1-z^2)}{(z^2+3z+1)^2} - \dfrac{4(z^3+z^2-4z-3)}{(z^2+3z+1)^3}$

(d) $Y(z) = \dfrac{8z^2}{(z^2+3z+1)^2}$

7.25 (a) $Y(z) = \dfrac{z}{z-3}$

(b) $Z(z) = \dfrac{z^6}{z-3}$

(c) $H(z) = \dfrac{z^3(9-2z)}{(z-3)^2}$

(d) $G(z) = -\dfrac{z^3(9-2z)}{(z-3)^2} + \dfrac{z^3[4z^2-69z+81]}{(z-3)^3}$

7.27 $p[n] = [1,-1,0,0,0,0,-1,1]$

7.29 (a) $x[n] = 2.309\sin 60^{\circ} nu[n]$

(b) $y[n] = -\dfrac{1}{2}(-1)^n u[n] + \dfrac{3}{2}u[n]$

(c) $h[n] = 8\left(-\dfrac{1}{2}\right)^n u[n] - 8\cos 120^{\circ} nu[n]$

7.31 (a) $x[n] = 1.2(3)^n u[n] - 0.2(0.5)^n u[n]$

(b) $y[n] = \dfrac{1}{6}\delta[n] - \dfrac{17}{3}(2)^n u[n] + \dfrac{37}{3}(3)^n u[n]$

7.33 (a) $(0.866)^n \cos(54.73^{\circ} n)u[n] - 0.7072(0.866)^n \sin(58.74^{\circ} n)u[n]$

(b) $(0.8)^n \cos(60^{\circ} n)u[n] + 2.02(0.8)^n \sin(60^{\circ} n)u[n]$

7.35 $y[n] = 2(0.6)^n \cos(146.4n)u[n] + 3.02(0.6)^n \sin(146.4n)u[n]$

7.37 $y[n] = 3(2)^n u[n] - 2(1.5)^n u[n]$

7.39 $y[n] = \frac{2}{3}u[n] - 0.1491\left(\frac{1}{\sqrt{6}}\right)^n \sin(114.1^\circ n)u[n]$

7.41 $H(z) = 2\delta[n] + \frac{3z}{z-0.6} - \frac{5z}{z+0.8}$

7.43 $Y(z) = \frac{z}{z-3}$

7.45 $(z+1)\left[\frac{1}{(z-1)^2} - \frac{3}{3z-1}\right]$

7.47 (a) $H(z) = \frac{1}{1+z^{-2}}$

(b) $y[n] + y[n-2] = x[n]$

7.49 $H(z) = 2 - \frac{1}{z} + \frac{1}{z^2} - \frac{z}{z+1}$

$h[n] = 2\delta[n] - \delta[n-1] + \delta[n-3] - (-1)^n u[n]$

7.51 (a) $X =$

```
1/((5*z)/4 – 1) + 1/2
```

(b) $X =$

```
(15*z)/(5*z – 3)^2
```

7.53 See Figure D.30.

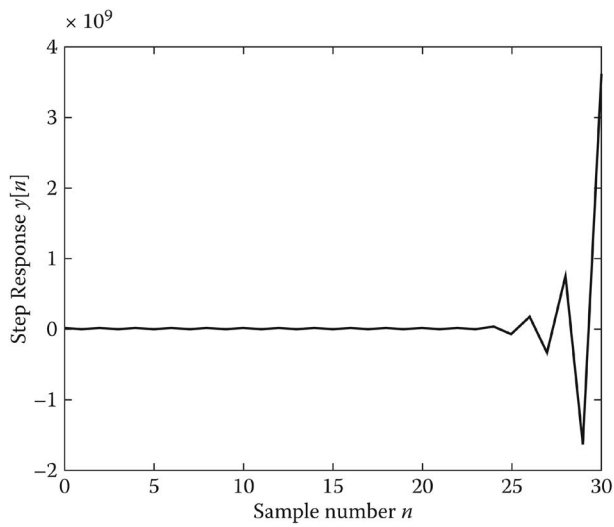


FIGURE D.30 For Prob. 7.53.

7.55 $z =$

$$0.6907 + 0.8958i$$
$$0.6907 - 0.8958i$$
$$-0.7815 + 0.0000i$$

$p =$

$$0.7500 + 0.6614i$$
$$0.7500 - 0.6614i$$
$$-1.0000 + 0.0000i$$
$$-1.0000 + 0.0000i$$

7.57 Unstable.

Index

A

ABET, 74
Accumulation, 307, 347
Amplitude modulation (AM), 267, 268
Amplitude transformation, 34

B

Band-limited signal, 272
BIBO stability, 104, 105, 167
Block diagram, 88, 98

C

Circuit analysis, 106, 163, 216, 264
Codes of ethics, 336
Communication, 75
Communication network, 295
Communication systems, 295
Complex poles, 147
Computing with MATLAB, 54, 101, 169, 221, 277, 320, 367
Conjugation, 303
Continuous-time signals, 3, 13
  operations on, 32
Control systems, 167, 243
Convolution, 74, 257, 348
  applications of, 104
  discrete-time type of, 90
  graphical type of, 81
Convolution integral, 76
  properties of, 78

D

Deconvolution, 99
Discrete Fourier transform (DFT), 295, 313
  applications of, 323
  properties of, 314
Discrete-time Fourier transform (DTFT), 295, 296
  pairs of, 310
  properties of, 301, 309
Discrete-time signal, 3, 4, 27
  operations on, 36
DSP systems, 53
Duality, 256

E

Electric circuits, 52
Even symmetry, 208
Expansion, 38
Exponential sequence 30
Exponential signal, 20

F

Fast Fourier transform (FFT), 319
Filters, 218, 219
Final value, 137, 348
Fourier, Jean Baptiste Joseph, 190
Fourier series, 190
  amplitude-phase form of, 201
  applications, 216
  definition of, 191, 192
  exponential type of, 199
  properties of, 205
  sine-cosine form of, 201
  trigonometric type of, 192
  truncated complex type of, 214
Fourier transform, 243
  applications of, 263
  definition of, 244, 245
  pairs of, 259
  properties of, 232, 258
Frequency convolution, 306
Frequency differentiation, 134, 255, 305
Frequency scaling, 345
Frequency shifting, 131, 254, 302

G

Global positioning system (GPS), 1

I

IEEE, 336
Impulse response, 75, 76
Initial value, 137, 348
Integrodifferential equation, 160
Inverse Laplace transform, 144, 262
Inverse z-transform, 355

L

Laplace, Pierre Simon, 123
Laplace transform, 123
  applications of, 160
  definition of, 124
  pairs of, 140
  properties of, 128, 139
Linear difference equation, 361

Linearity, 128, 205, 252, 301, 344
Long division expansion, 355

M

Modulation, 136, 346

O

Odd symmetry, 208

P

Parseval relation, 308
Parseval theorem, 209, 273, 274
Partial fraction expansion, 357
Properties of convolution sum, 91, 92

R

Rectangular pulse function, 18
Region of convergence, 125, 341
Repeated poles, 146

S

Sampling, 270
Sampling theorem, 218
Scaling, 129
Signals, 2, 3
  analog type of, 6
  classification of, 3
  continuous-time type of, 3, 13
  digital type of, 6
  discrete-time type of, 3, 4, 27
  energy type of, 7, 8
  even type of, 8
  nonperiodic type of, 4
  odd type of, 8
  periodic type of, 4, 5, 135, 192
  power type of, 7
Simple poles, 145
Sinusoidal sequence, 29
Sinusoidal signal, 20
Spectrum analyzers, 217, 218
Square-law devices, 52
Systems, 2, 4
  causal type, 46, 47
  classification of, 43
  continuous-time type of, 46
  discrete-time type of, 46
  linear type of, 47, 48
  noncausal type, 46
  nonlinear type of, 47
  time-invariant type of, 48
  time-varying type of, 48
  with memory, 50
  without memory, 50

T

Time convolution, 132, 306
Time differentiation, 131, 254
Time integration, 133, 255
Time periodicity, 135
Time reversal, 32, 38, 207, 303, 346
Time scaling, 33, 207, 252, 304
Time shifting, 34, 38, 130, 206, 253, 302, 344
Touch-tone telephone, 323
Transfer function, 155, 156, 363
Triangular pulse function, 19

U

Unit impulse function, 15
Unit impulse sequence, 28
Unit ramp function, 17
Unit ramp sequence, 28
Unit step function, 13
Unit step sequence, 27

W

Windowing, 324

Z

z-transform, 336
  applications of, 360
  definition of, 337
  pairs of, 351
  properties of, 343, 350